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INVERSE CONDUCTIVITY PROBLEM WITH INTERNAL DATA*

Faouzi Triki[1)]
Laboratoire Jean Kuntzmann, UMR CNRS 5224, Université Grenoble-Alpes, 700 Avenue Centrale, 38401 Saint-Martin- d'Hères, France
Email: faouzi.triki@univ-grenoble-alpes.fr
Tao Yin
Institute of Computational Mathematics and Scientific/Engineering Computing, Academy of Mathematics and Systems Science, Chinese Academy of Science, Beijing 100190, China
Email: yintao@lsec.cc.ac.cn

Abstract

This paper concerns the reconstruction of a scalar coefficient of a second-order elliptic equation in divergence form posed on a bounded domain from internal data. This problem finds applications in multi-wave imaging, greedy methods to approximate parameter-dependent elliptic problems, and image treatment with partial differential equations. We first show that the inverse problem for smooth coefficients can be rewritten as a linear transport equation. Assuming that the coefficient is known near the boundary, we study the well-posedness of associated transport equation as well as its numerical resolution using discontinuous Galerkin method. We propose a regularized transport equation that allow us to derive rigorous convergence rates of the numerical method in terms of the order of the polynomial approximation as well as the regularization parameter. We finally provide numerical examples for the inversion assuming a lower regularity of the coefficient, and using synthetic data.

Mathematics subject classification: 35R30, 65N21.
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1. Introduction

Let Ω be a C^6-smooth bounded domain of $\mathbb{R}^n$, $n = 2, 3$, with boundary Γ. Let $\nu(x)$ be the outward normal vector at $x \in \Gamma$, and $d = \sup_{x,y\in\Omega} \|x - y\|$ be the diameter of Ω. We set, for $\eta \in (0, d)$, $\sigma_0 \in W^{2,\infty}(\Omega)$, $\Omega_\eta = \{x \in \Omega, \mathrm{dist}(x, \Gamma) > \eta\}$, and $0 < k_1 < k_2$,

$$\Sigma = \{\sigma \in W^{2,\infty}(\Omega);\ \sigma = \sigma_0 \ \text{ in } \Omega \setminus \overline{\Omega_\eta},\ k_1 \le \sigma,\ \|\sigma\|_{W^{2,\infty}(\Omega)} \le k_2\}.$$

Let g be fixed in $H^{\frac{7}{2}}(\Gamma)$, and satisfy $\int_\Gamma g dx = 0$. Then, according to the classical elliptic regularity theory

$$\mathrm{div}(\sigma\nabla u) = 0 \text{ in } \Omega, \qquad \sigma\partial_\nu u = g \text{ on } \Gamma,\ \int_\Omega u_\sigma dx = 0, \tag{1.1}$$

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1) Corresponding author

has a unique solution $u_\sigma \in H^5(\Omega)$ [1], and there exists a constant $c = c(\Sigma, \Omega) > 0$ such that

$$\|u_\sigma\|_{H^5(\Omega)} \leq c. \tag{1.2}$$

The goal of this work is to study the following inverse problem (IP): Given σ_0 and the interior data $u_\sigma|_\Omega$, to reconstruct the conductivity $\sigma|_\Omega$.

This inverse problem is of importance in many different scientific and engineering fields including photoacoustic tomography, studies of effective properties of composite materials, and approximation of parametric partial differential equations. Photoacoustic tomography is a recent hybrid imaging modality that couples diffusive optical waves with ultrasound waves to achieve high-resolution imaging of optical properties of biological tissues [2–7]. The inverse problem (IP) appears in the second inversion, called quantitative photoacoustic tomography, where the derived internal data is used to recover the optical coefficients of the sample [8, 9]. Motivated by the search for sharp bounds on the effective moduli of composites many researchers have considered the problem of characterizing mathematically among all the gradient fields those solving the equation (1.1) for some function σ within the set Σ. In the context of approximation of parameter-dependent elliptic problems by greedy algorithms the inverse problem (IP) has been considered with infinitely many interior data available [10]. Hence solving the inverse problem with a single datum may reduce the dimensionality of the set of parameters used to accurately approximate a targeted compact set of solutions [11].

Given σ_0 and the interior data $u_\sigma|_\Omega$, the inverse problem can be recasted as a linear steady transport equation satisfied by $\sigma \in \Sigma$,

$$\nabla\sigma \cdot \nabla u_\sigma + (\Delta u_\sigma)\sigma = 0 \text{ in } \Omega.$$

The steady transport equation is one of the basic equations in mathematical physics. It is widely used in fluid mechanics, for example to model mass transfer [12]. From the mathematical point of view there are several results addressing the well-posdeness of the equation. In order to briefly review some of these results we introduce suitable boundary conditions. To do so we split the boundary of Γ into three disjoint parts, the inflow Γ_{in}, the outflow set Γ_{out}, and the characteristic set Γ_0, defined by

$$\Gamma_{\text{in}} = \{x \in \Gamma : \nabla u_\sigma \cdot \nu < 0\}, \quad \Gamma_{\text{out}} = \{x \in \Gamma : \nabla u_\sigma \cdot \nu > 0\}, \quad \Gamma_0 = \Gamma \setminus (\Gamma_{\text{in}} \cup \Gamma_{\text{out}}). \tag{1.3}$$

Assuming that ∇u_σ never vanishes in Ω and using the method of characteristics, one can easily show that the system

$$\nabla\sigma \cdot \nabla u_\sigma + (\Delta u_\sigma)\sigma = 0 \quad \text{in } \Omega, \qquad \sigma = \sigma_0 \quad \text{on } \Gamma_{\text{in}}, \tag{1.4}$$

admits a unique solution. The method of characteristics can not be applied when the set of characteristic curves has a complex structure, for example when ∇u_σ vanishes. In order to overcome this difficulty, many works have considered the case where the lower order term dominates the transport term. In this framework the theory of linear steady transport equations becomes part of a more general theory of degenerate elliptic equations ([13–15], see also Chapter 12 in [12] and references therein). Let $\kappa > 0$ be a fixed constant. When $n = 3$, and assuming that the interior data u_σ verifies

$$\inf_{x\in\Omega} |\Delta u_\sigma(x)| > \kappa > 0, \tag{1.5}$$

it was proved in [15], by using the vanishing viscosity method, that the system (1.4) admits a weak solution in $L^2(\Omega)$, satisfying

$$\|\sigma\|_{L^2(\Omega)} \leq \frac{1}{\kappa}\|\nabla\sigma_0 \cdot \nabla u_\sigma + (\Delta u_\sigma)\sigma_0\|_{L^2(\Omega)}.$$

If, in addition, $\partial\Gamma_{\text{in}}$ is a one-dimensional C^1 manifold, then there is a unique weak solution $\sigma \in L^2(\Omega)$. However in general without geometrical assumptions on the characteristic set, the system (1.4) may have other weak solutions. Indeed the problems of uniqueness and regularity of solutions to the system (1.4) are difficult issues, most of the available results relate to the case of multi-connected domains with isolated inflow boundary Γ_{in} [14, 15]. In [16], the author assuming that $|\Delta u_\sigma(x)| \neq 0$ or $|\nabla u_\sigma(x)| \neq 0$ in Ω has derived Lipschitz stability estimates for the inversion.

Linear steady transport equations is also part of a general theory developed by Friedrichs dealing with symmetric positive systems of first-order linear partial differential equations [17]. Recently, numerical methods based on discontinuous Galerkin methods have renewed interest in Friedrichs systems [18–20]. In this settings the conditions required for existence and uniqueness of solutions to the system are almost similar to the ones derived in [13–15].

Due the specificity of our equation (the lower order term is linked to the transport speed) the condition (1.5) which is the minimal requirement for the existence of solutions in all the presented variational approaches is stronger than assuming that ∇u_σ does not vanish in Ω. Furthermore, if the interior data is recovered with zero noise the existence of solution is in fact guaranteed from the fact that the transport equation is originated from the elliptic system (1.1). The uniqueness of solutions to the system (2.1) is established in [21] using elliptic unique continuation properties of the transport speed ∇u_σ. However, in applications, the internal data $u_\sigma|_\Omega$ is usually measured with limited accuracy. Hence, studying the well-posedness of the inverse problem (IP) when the measurements are noisy is of critical importance.

We aim in this paper to study the well-posedness of the inverse problem (IP) as well as its numerical resolution. If the interior measurement is noisy, that is, only $U^\delta \in H^5(\Omega)$ is available, satisfying

$$\|U^\delta - u_\sigma\|_{H^5(\Omega)} \leq \delta\|u_\sigma\|_{H^5(\Omega)}, \qquad \int_\Omega U^\delta dx = 0, \tag{1.6}$$

with $\delta \in (0, 1)$ is small enough, could we construct $\sigma_\delta \in L^2(\Omega)$, solution to the transport equation (1.4), with U^δ substituting u_σ, that tends to $\sigma \in \Sigma$ when the noise δ tends to zero? This question is also related to the numerical approximation of the solution to the inverse problem (IP).

Since in general a noisy data U^δ does not belong to $\mathcal{U} := \{u_\sigma \in H^5(\Omega) : \sigma \in \Sigma\}$, the transport equation (2.1) with u_σ substituted by U^δ does not need to have a solution (see example 2.4 in [9]). Therefore, it is necessary to find an adequate regularization method to the linear equation (2.1). The vanishing viscosity method is a well known way to regularize this type of equation, it consists in adding a second order term $\varepsilon\Delta\cdot$, with ε is a small positive parameter [21]. The regularized equation becomes then an elliptic one and the singularities of targeted function σ may not be visible in the regularized solution. In this paper we propose a first order regularization method based on an original regularization of the lower order term that does not improve the regularity of the solution. We note that since the transport equation (1.4) is solved in the weak L^2 topology, the assumption on the regularity of σ, u_σ, and the noisy data U^δ may be significantly reduced. Indeed, we considered such a large smoothness to ease the presentation of our theoretical stability estimates. Note that the regularity of the domain as well as the regularity of the data are chosen to be large to ease the stability and convergence analysis. The obtained results can be easily applied to less regular domains and boundary data by using interpolation inequalities.

The paper is organized as follows. In section 2, we study the stability of the solution of the inverse problem (IP) with respect to the noise in the internal measurement. We prove the existence and uniqueness of solution to the regularized transport equation (2.4) in Theorem 2.1. We also show the convergence of the regularized solution to the targeted conductivity distribution in Theorem 2.2. Section 3 is devoted to the numerical approximation of the solution using discontinuous Galerkin method. We derive error estimates of the numerical approximation in Theorem 3.1. We finally provide in section 4 several numerical tests based on synthetic data, and assuming a lower regularity of the conductivity distribution.

2. Stability Estimates

In this section we study the well-posdeness of the regularized transport equation using a variational approach developed in [17]. We introduce the following modified transport equation

$$2\nabla\gamma \cdot \nabla u_\sigma + \gamma \Delta u_\sigma = 0 \quad \text{in } \Omega \qquad \text{and} \quad \gamma = \gamma_0, \tag{2.1}$$

for $\gamma \in H^1(\Omega)$ and $\gamma_0 = \sqrt{\sigma_0}$. We know from the system (1.1) that $\gamma = \sqrt{\sigma}$ is a solution of (2.1). We deduce from [5, 21] that $\sqrt{\sigma}$ is indeed the unique solution.

Let $U^\delta \in H^5(\Omega)$ be given, and satisfying (1.6). Our first objective in this section is to derive for $\delta > 0$, small enough an approximate solution of the following noisy transport equation

$$2\nabla\gamma_\delta \cdot \nabla U^\delta + \gamma_\delta \Delta U^\delta = 0 \quad \text{in } \Omega \qquad \text{and} \quad \gamma = \gamma_0, \tag{2.2}$$

Since $\gamma_\delta = \gamma_0$ and U^δ are given in Ω_η^C, we can modify the behavior of U^δ in order to have a right hand term in $L^2(\Omega)$, and $\sigma_0 \partial_\nu U^\delta = g$ on Γ, with g is chosen such that the associated inflow and outflow boundaries be well-separated, namely,

$$\text{dist}(\Gamma_{\text{in}}, \Gamma_{\text{out}}) := \min_{(x,y)\in\Gamma_{\text{in}}\times\Gamma_{\text{out}}} |x - y| > 0. \tag{2.3}$$

This condition is necessary and sufficient to define traces of functions belonging to the graph space of the steady transport unbounded operator in [18]

$$L^2(|g|;\Gamma) := \left\{ v \text{ measurable on } \Gamma : \int_\Gamma |g| v^2 ds(x) < \infty \right\}.$$

Since the transport equation (2.2) does not fall within the classical variational framework to prove the existence, uniqueness of solutions, we introduce an auxiliary problem indexed by $\varepsilon > 0$ which should be small enough.

Fix $\varepsilon \in (0, 1)$, and define the regularized system corresponding to (2.2) as follows

$$\beta \cdot \nabla\gamma + \mu_\varepsilon \gamma = 0 \text{ in } \Omega \tag{2.4}$$

where $\beta = \nabla U^\delta$ and $\mu_\varepsilon = \frac{1}{2}\Delta U^\delta + \varepsilon$. Due to the regularity of $U^\delta \in H^5(\Omega)$, the speed β lies in $C^{1,1}(\overline{\Omega})$, with $\|\nabla\beta_i\|_{L^\infty(\Omega)^2} \leq L_i < \infty$ for $i = 1, 2$, $\beta_{,i}$ being the components of β. We deduce from (1.6) that the Lipschitz constant $L := \max_{i=1,2} L_i$ only depends on g, Σ and Ω.

Next we prove the existence and uniqueness of (2.4) with inflow boundary condition

$$\gamma = \gamma_0 \text{ on } \Gamma_{\text{in}}.$$

Before that, we introduce the Graph space $V := \{v \in L^2(\Omega) | \beta \cdot \nabla v \in L^2(\Omega)\}$ equipped with the natural scalar product

$$(v, w)_V := (v, w)_{L^2(\Omega)} + (\beta \cdot \nabla v, \beta \cdot \nabla w)_{L^2(\Omega)}, \quad \forall v, w \in V,$$

and the norm $\|v\|_V := ((v, v)_V)^{1/2}$. It follows that V is a Hilbert space and the triple $\{V, L^2(\Omega), V'\}$ is a Gelfand triple [18]. Denote

$$L^2(|\beta \cdot n|, \Gamma) := \left\{ v \text{ is measurable on } \Gamma | \int_\Gamma |\beta \cdot \nu| v^2 ds < \infty \right\}$$

the trace space. Lemma .1 in the appendix allows us to define traces of functions belonging to the graph space V and to use an integration by parts formula. In addition, for a real number r, we define its positive and negative parts respectively as

$$r^{\oplus} := \frac{1}{2}\{|r| + r\}, \qquad x^{\ominus} := \frac{1}{2}\{|r| - r\}.$$

We consider the weak form of (2.4) as follows:

$$\text{Find } \gamma_\varepsilon^\delta \in V \text{ such that } a(\gamma_\varepsilon^\delta, w) = \int_\Gamma (\beta \cdot \nu)^{\ominus} \gamma_0 w \, ds \text{ for all } w \in V, \tag{2.5}$$

where

$$a(v, w) := \int_\Omega (\beta \cdot \nabla v + \mu_\varepsilon v) \, w \, dx + \int_\Gamma (\beta \cdot \nu)^{\ominus} v w \, ds.$$

Theorem 2.1. *Let g be fixed in $H^{\frac{7}{2}}(\Gamma)$ satisfying $\int_\Gamma g dx = 0$ and condition (2.3). Then there exists a unique solution $\gamma_\varepsilon^\delta \in V$ to the system (2.4) for $\varepsilon \in (0, 1)$.*

Proof. We follow the proof in [18, 20] and trace out the dependence of constant in terms of the regularized parameter ε. The proof proceeds in four steps. Further $c > 0$ denotes a generic constant that only depends on Σ, g and Ω.

We first prove that (2.5) admits at most one solution. For all $v \in V$, we obtain from integration by parts that

$$\begin{aligned} a(v, v) &= \varepsilon \int_\Omega v^2 \, dx + \int_\Gamma \left[\frac{1}{2}(\beta \cdot \nu) + (\beta \cdot \nu)^{\ominus} \right] v^2 \, ds \\ &\geq \|v\|^2_{L^2(\Omega)} \varepsilon + \frac{1}{2} \int_\Gamma |g| v^2 \, ds, \end{aligned} \tag{2.6}$$

which implies the desired uniqueness. To prove the existence, we introduce an auxiliary problem:

$$\text{Find } v' \in V \text{ such that } a(v', w) = -a_0(\gamma_0, w) \text{ for all } w \in V, \tag{2.7}$$

where

$$a_0(v, w) := \int_\Omega (\beta \cdot \nabla v + \mu_\varepsilon v) \, w \, dx.$$

The map $V \ni w \mapsto a_0(\gamma_0, w) \in \mathbb{R}$ is bounded in $L^2(\Omega)$. Due to the regularity of γ_0, the function $f := \beta \cdot \nabla\gamma_0 + \mu_\varepsilon \gamma_0$ lies in $L^2(\Omega) \subset V'$. Then if (2.7) admits a solution $v' \in V$, we can obtain that $\gamma_\varepsilon^\delta = v' + \gamma_0 \in V$ satisfies (2.5) which gives the existence.

Hence it remains to prove that (2.7) is well-posed. The uniqueness of (2.7) also follows from (2.6). Set $V_0 := \{v \in V : v|_{\Gamma_{\text{in}}} = 0\}$, and consider the following problem:

$$\text{Find } \ v_0 \in V_0 \ \text{ such that; } a_0(v_0, w) = \int_\Omega f w\, dx \ \text{ for all } \ w \in L^2(\Omega). \tag{2.8}$$

If (2.8) is well-posed and let $v_0 \in V_0$ be its unique solution, we have that $v_0 \in V$ and $a(v_0, w) = a_0(u, w)$ for all $w \in V$. It means that v_0 is a solution of (2.7). It remains to prove that (2.8) is well-posed.

Here we shall apply the Banach-Nečas-Babuška (BNB) Theorem .1 with $X = V_0$ and $Y = L^2(\Omega)$. In fact V_0 is a Hilbert space since V_0 is closed in V and $L^2(\Omega)$ is a reflexive Banach space. The right-hand side in (2.8) is a bounded linear form in $L^2(\Omega)$ and $a_0 \in \mathcal{L}(V_0 \times L^2(\Omega), \mathbb{R})$ since

$$|a_0(v, w)| \leq \left(1 + \frac{1}{2}\|\Delta U^\delta\|_{L^\infty(\Omega)}\right)\|v\|_V \|w\|_{L^2(\Omega)}, \quad \forall (v, w) \in X \times Y.$$

It remains to verify the conditions (.1) and (.2) of the BNB Theorem.

(i). Proof of condition (.1). For $v \in V_0$, we set

$$\mathcal{S}_v := \sup_{0 \neq w \in L^2(\Omega)} \frac{a_0(v, w)}{\|w\|_{L^2(\Omega)}}.$$

Similar as (2.6), we can get

$$a_0(v, v) \geq \varepsilon \|v\|^2_{L^2(\Omega)} \quad \text{for} \quad v \in V_0.$$

Then for $0 \neq v \in V_0$,

$$\|v\|_{L^2(\Omega)} \leq \frac{1}{\varepsilon} \frac{a_0(v, v)}{\|v\|_{L^2(\Omega)}} \leq \frac{1}{\varepsilon}\mathcal{S}_v.$$

On the other hand,

$$\begin{aligned}
\|\beta \cdot \nabla v\|_{L^2(\Omega)} &= \sup_{0 \neq w \in L^2(\Omega)} \frac{\int_\Omega (\beta \cdot \nabla v) w\, dx}{\|w\|_{L^2(\Omega)}} \\
&= \sup_{0 \neq w \in L^2(\Omega)} \frac{a_0(v, w) - \int_\Omega (\frac{1}{2}\Delta U^\delta + \varepsilon) v w\, dx}{\|w\|_{L^2(\Omega)}} \\
&\leq \mathcal{S}_v + (\frac{1}{2}\|\Delta U^\delta\|_{L^\infty(\Omega)} + 1)\|v\|_{L^2(\Omega)} \\
&\leq \left(1 + (\frac{1}{2}\|\Delta U^\delta\|_{L^\infty(\Omega)} + 1)\varepsilon^{-1}\right)\mathcal{S}_v.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|v\|_V^2 &= \|v\|^2_{L^2(\Omega)} + \|\beta \cdot \nabla v\|^2_{L^2(\Omega)} \\
&\leq \left[\varepsilon^{-2} + \left(1 + (\frac{1}{2}\|\Delta U^\delta\|_{L^\infty(\Omega)} + 1)\varepsilon^{-1}\right)^2\right]\mathcal{S}_v^2 \leq c\varepsilon^{-2}\mathcal{S}_v^2.
\end{aligned}$$

Then, we can take $C_{sta} := c\varepsilon$, whence we infer condition (.1).

(ii). Proof of condition (.2). Let $w \in L^2(\Omega)$ be such that $a_0(v,w) = 0$ for all $v \in V_0$. Taking $v \in C_0^\infty(\Omega)$ first we get $\mu_\varepsilon w - \operatorname{div}(\beta w) = 0$ in Ω. Hence, $\beta \cdot \nabla w = \mu_\varepsilon w - (\operatorname{div}\beta) w \in L^2(\Omega)$ implying that $w \in V$. Then we have, for all $v \in V_0$

$$\begin{aligned}\int_\Gamma (\beta\cdot\nu) vw &= \int_\Omega [(\beta\cdot\nabla v)w + (\beta\cdot\nabla w)v + (\nabla\cdot\beta)vw]\,dx\\ &= a_0(v,w) = 0.\end{aligned}$$

Since Γ_{in} and Γ_{out} are well-separated, there exist two functions $\psi^\pm \in C^\infty(\overline{\Omega})$ such that [18]

$$\psi^{\mathrm{out}} + \psi^{\mathrm{in}} = 1 \quad \text{in} \quad \overline{\Omega}, \qquad \psi^{\mathrm{in}}|_{\Gamma_{\mathrm{out}}} = 0, \quad \psi^{\mathrm{out}}|_{\Gamma_{\mathrm{in}}} = 0.$$

Taking $v = \psi^{\mathrm{out}} w \in V_0$, we obtain that

$$0 = \int_\Gamma (\beta\cdot\nu)\psi^{\mathrm{out}} w^2\,ds = \int_{\Gamma_{\mathrm{in}}} (\beta\cdot\nu) w^2\,ds = \int_{\Gamma_{\mathrm{out}}} (\beta\cdot\nu)^{\oplus} w^2\,ds,$$

which further implies that $w = 0$ on Γ_{out}. Finally, we observe that

$$\begin{aligned} 0 &= \int_\Omega [\mu_\varepsilon - \operatorname{div}(\beta w)] w\,dx\\ &= \varepsilon \int_\Omega w^2\,dx + \frac{1}{2}\int_\Gamma (\beta\cdot\nu) w^2\,ds\\ &= \varepsilon \|w\|^2_{L^2(\Omega)}.\end{aligned}$$

Therefore, $w = 0$ in Ω, which completes the proof. □

The unique solution of (2.1) and the regularized one (2.4), satisfy the following stability estimate.

Theorem 2.2. *Let g be fixed in $H^{\frac{7}{2}}(\Gamma)$ satisfying $\int_\Gamma g dx = 0$ and condition* (2.3)*. Then, there exist $c > 0$ and $s \in (0, \frac{1}{2})$, that only depends on $\sigma, g, \eta,$ and Ω such that*

$$\int_{\Omega_\eta} |(\gamma_\varepsilon^\delta)^2 - \gamma^2|^{1/2}\,dx \le c\left(1 + \frac{\delta}{\varepsilon}\right)(\varepsilon + \delta)^s, \tag{2.9}$$

where γ and $\gamma_\varepsilon^\delta$ are the solutions respectively to (2.1) *and* (2.4)*. If in addition $|\nabla u_\sigma|$ does not vanish in Ω_η, the inequality* (2.9) *holds with $s = \frac{1}{2}$.*

Proof. Further $c > 0$ denotes a generic constant that only depends on σ, g, η, and Ω. We deduce from equations (2.1) and (2.4) that $\xi = \gamma_\varepsilon^\delta - \gamma$ solves

$$\beta\cdot\nabla\xi + \mu_\varepsilon \xi = \nabla(u_\sigma - U^\delta)\cdot\nabla\gamma + \left(\frac{1}{2}\Delta(u_\sigma - U^\delta) - \varepsilon\right)\gamma \quad \text{in } \Omega \qquad \text{and} \quad \xi = 0 \quad \text{on } \Gamma_{\mathrm{in}}.$$

Using the variational formulation for the transport equation, and the fact that $\sigma \in \Sigma$, we find

$$\varepsilon \|\xi\|_{L^2(\Omega)} \le c\|U_\delta - u_\sigma\|_{H^1(\Omega)} + c\|U^\delta - u_\sigma\|_{H^2(\Omega)} + c\varepsilon.$$

Combining the inequality above with estimate (1.6), we get

$$\|\xi\|_{L^2(\Omega)} \le c\left(\frac{\delta}{\varepsilon} + 1\right). \tag{2.10}$$

Since $\sigma \in \Sigma$, we immediately deduce from (2.10) the following bound

$$\|\gamma_\varepsilon^\delta\|_{L^2(\Omega)} \le c\left(\frac{\delta}{\varepsilon}+1\right). \tag{2.11}$$

Recall that γ^2 and $(\gamma_\varepsilon^\delta)^2$ solve respectively the following systems:

$$\operatorname{div}\left((\gamma_\varepsilon^\delta)^2\nabla U^\delta\right) = -2\varepsilon(\gamma_\varepsilon^\delta)^2 \quad \text{in } \Omega \qquad \text{and} \quad (\gamma_\varepsilon^\delta)^2 = \sigma_0 \quad \text{on } \Gamma,$$

and

$$\operatorname{div}(\gamma^2\nabla u_\sigma) = 0 \quad \text{in } \Omega \qquad \text{and} \quad \gamma^2 = \sigma_0 \quad \text{on } \Gamma,$$

Let sgn_0 be the sign function defined on $\mathbb{R}$ by: $\mathrm{sgn}_0(t) = -1$ if $t < 0$, $\mathrm{sgn}_0(0) = 0$ and $\mathrm{sgn}_0(t) = 1$ if $t > 0$. Note that

$$\begin{aligned}\operatorname{div}\left(|(\gamma_\varepsilon^\delta)^2 - \gamma^2|\nabla U^\delta\right) &= \mathrm{sgn}_0((\gamma_\varepsilon^\delta)^2-\gamma^2)\operatorname{div}\left(((\gamma_\varepsilon^\delta)^2-\gamma^2)\nabla U^\delta\right)\\ &= \mathrm{sgn}_0((\gamma_\varepsilon^\delta)^2-\gamma^2)\left[\operatorname{div}\left(\gamma^2\nabla(u_\sigma - U^\delta)\right) - 2\varepsilon(\gamma_\varepsilon^\delta)^2\right].\end{aligned}$$

We get by integrating by parts that

$$\begin{aligned}\|\left((\gamma_\varepsilon^\delta)^2-\gamma^2\right)|\nabla U^\delta|^2\|_{L^1(\Omega)} &= \int_\Omega \left|\operatorname{div}(|(\gamma_\varepsilon^\delta)^2-\gamma^2|\nabla U^\delta)\right||U^\delta|\,dx\\ &\le \int_\Omega \left(\left|\operatorname{div}\left(\gamma^2\nabla(u_\sigma-U^\delta)\right)\right| + 2\varepsilon(\gamma_\varepsilon^\delta)^2\right)|U^\delta|\,dx.\end{aligned}$$

Thus,

$$\begin{aligned}&\|\left((\gamma_\varepsilon^\delta)^2-\gamma^2\right)|\nabla u_\sigma|^2\|_{L^1(\Omega)}\\ &\le \|((\gamma_\varepsilon^\delta)^2-\gamma^2)(|\nabla u_\sigma|^2 - |\nabla U^\delta|^2)\|_{L^1(\Omega)} + c\|u_\sigma - U^\delta\|_{H^2(\Omega)} + c(\varepsilon+\delta).\end{aligned}$$

We obtain that

$$\||\left((\gamma_\varepsilon^\delta)^2-\gamma^2\right)|\nabla u_\sigma|^2\|_{L^1(\Omega)} \le c(\varepsilon+\delta)(1+\frac{\delta^2}{\varepsilon^2}). \tag{2.12}$$

Denote $\Omega_\eta^t := \{x \in \Omega_\eta : |\nabla u_\sigma(x)| \ge t\}$. When t tends to zero we expect $|\Omega_\eta \setminus \Omega_\eta^t|$ to approach zero. The rate of decay depends on how does $|\nabla u_\sigma(x)|$ vanish at its critical points. The proof of this technical lemma is given in Appendix 4.

Lemma 2.1. *Let $\Omega_\eta^t := \{x \in \Omega_\eta : |\nabla u_\sigma(x)| \ge t\}$. Then the following inequality holds*

$$0 \le |\Omega_\eta \setminus \overline{\Omega_\eta^t}| \le ct^\alpha, \tag{2.13}$$

where $c > 0$ and $\alpha > 0$ only depend on σ, Ω, η and g.

Then, we get

$$\begin{aligned}&\int_{\Omega_\eta^t} |\gamma^2 - (\gamma_\varepsilon^\delta)^2|^{1/2}\,dx\\ &\le t^{-1}\int_{\Omega_\eta^t} |\gamma^2 - (\gamma_\varepsilon^\delta)^2|^{1/2}|\nabla u_\sigma|\,dx \le t^{-1}\int_\Omega |\gamma^2-(\gamma_\varepsilon^\delta)^2|^{1/2}|\nabla u_\sigma|\,dx\\ &\le t^{-1}|\Omega|^{1/2}\left(\int_\Omega |\gamma^2 - (\gamma_\varepsilon^\delta)^2||\nabla u_\sigma|^2\,dx\right)^{1/2} \le ct^{-1}\left((\varepsilon+\delta)(1+\frac{\delta^2}{\varepsilon^2})\right)^{1/2},\end{aligned}$$

and

$$\int_{\Omega_\eta\backslash\overline{\Omega_\eta^t}} |\gamma^2-(\gamma_\varepsilon^\delta)^2|^{1/2}\,dx$$
$$\leq |\Omega_\eta\backslash\Omega_\eta^t|^{1/2}\|\gamma^2-(\gamma_\varepsilon^\delta)^2\|_{L^1(\Omega)} \leq C|\Omega_\eta\backslash\Omega_\eta^t|^{1/2}\left(\frac{\delta}{\varepsilon}+1\right) \leq ct^{\frac{\alpha}{2}}\left(\frac{\delta}{\varepsilon}+1\right).$$

Hence

$$\begin{aligned}
&\int_{\Omega_\eta\backslash\overline{\Omega_\eta^t}} |\gamma^2-(\gamma_\varepsilon^\delta)^2|^{1/2}\,dx \\
&\leq \int_{\Omega\backslash\Omega_\eta^t} |\gamma^2-(\gamma_\varepsilon^\delta)^2|^{1/2}\,dx + + \int_{\Omega_\eta^t} |\gamma^2-(\gamma_\varepsilon^\delta)^2|^{1/2}\,dx \\
&\leq c\left[t^{-1}\left((\varepsilon+\delta)(1+\frac{\delta^2}{\varepsilon^2})\right)^{1/2} + t^{\frac{\alpha}{2}}\left(\frac{\delta}{\varepsilon}+1\right)\right] \\
&\leq c\left[t^{-1}(\varepsilon+\delta)^{1/2} + t^{\frac{\alpha}{2}}\right]\left(\frac{\delta}{\varepsilon}+1\right).
\end{aligned}$$

Minimizing the right-hand side with respect to $t>0$, for fixed δ and ε in $(0,1)$, we find that the minimum is reached at $t=(\varepsilon+\delta)^{\frac{1}{\alpha+2}}$, and verifies

$$t^{-1}(\varepsilon+\delta)^{1/2} = t^{\frac{\alpha}{2}}.$$

Then, we get

$$\int_{\Omega_\eta} |\gamma^2-(\gamma_\varepsilon^\delta)^2|^{1/2}\,dx \leq c(1+\frac{\delta}{\varepsilon})(\varepsilon+\delta)^s, \quad s=\frac{\alpha}{2(\alpha+2)}, \tag{2.14}$$

which finishes the proof of the Theorem when $\Omega_\eta\setminus\overline{\Omega_\eta^t}$ is not empty. □

Assuming now that $|\nabla u_\sigma|$ does not vanish in Ω_η. Regarding the regularity of $|\nabla u_\sigma|$, $\Omega_\eta\setminus\overline{\Omega_\eta^t}$ is empty for $t>0$ small enough, that is $\Omega_\eta=\Omega_\eta^t$. We then deduce from (2.14) that the inequality (2.9) is valid with $s=\frac{1}{2}$.

Remark 2.1. Notice that $\varphi\to\int_\Omega|\varphi|^{1/2}dx$ defines a complete metric on $L^{1/2}(\Omega)$. In fact it is only a quasi-norm since it does not satisfy the triangle inequality. Meanwhile the Hölder inequality still holds [22].

Next, we study the regularity of the unique weak solution $\gamma_\varepsilon^\delta$ of the regularized transport equation (2.4). The main difficulty of the theory of the boundary value problem for the transport equation in the case of nonempty set $\partial\Gamma_{\rm in}$ is that a solution may develop singularities at $\partial\Gamma_{\rm in}$ (See for instance example 12.2.1 in [12]).

Theorem 2.3. *Let g be fixed in $H^{\frac{7}{2}}(\Gamma)$ satisfying $\int_\Gamma g dx=0$ and condition (2.3), and let $\gamma_\varepsilon^\delta\in V$ be the unique solution to (2.4) for $\varepsilon\in(0,1)$. Assume in addition that u_σ is a convex function. Then there exists $c_0>0$ that only depends on Σ, g and Ω such that if $\frac{\delta}{\varepsilon}\in(0,c_0)$, $\gamma_\varepsilon^\delta$ lies in $H^1(\Omega)$, and it satisfies*

$$\|\gamma_\varepsilon^\delta\|_{H^1(\Omega)} \leq c\varepsilon^{-1}, \tag{2.15}$$

where $c>0$ only depends on Σ, g and Ω.

Proof. Differentiating the regularized equation (2.4), we obtain that $\zeta = \nabla \gamma_\varepsilon^\delta$ satisfies the following Friedirich system

$$(\partial_k U^\delta I_n)\partial_k \zeta + \left((\frac{1}{2}\Delta U^\delta + \varepsilon) I_n + \mathcal{H}(U^\delta) \right) \zeta = -\frac{1}{2}\gamma_\varepsilon^\delta \nabla \Delta U^\delta,$$

∂_i denotes the partial derivative with respect to x_i, I_n is identity matrix in $\mathbb{R}^n$, $H(U_\varepsilon^\delta)$ is the Hessian matrix of U_ε^δ.

Since the function u_σ is convex, we deduce from (1.6) that there exists a constant $c_0 > 0$, that only depends on Σ, g and Ω, such that

$$\mathcal{H}(U^\delta) \geq -\frac{\delta}{2c_0} I_n.$$

Hence for $\frac{\delta}{\varepsilon} \in (0, c_0)$, we have

$$\varepsilon I_n + \mathcal{H}(U^\delta) \geq \frac{\varepsilon}{2}.$$

The variational necessary condition for the existence and uniqueness of solution to the Friedirich system is then satisfied, and we have [18]

$$\|\zeta\|_{(L^2(\Omega))^n} \leq c\varepsilon^{-1} \|\gamma_\varepsilon^\delta\|_{L^2(\Omega)}.$$

Combining the previous estimate with (2.11), gives the desired result. $\square$

Remark 2.2. Note that the assumption on the convexity of u_σ immediately implies $\Delta u_\sigma \geq 0$ since $\Delta u_\sigma = \mathrm{tr}(\mathcal{H}(u_\sigma))$.

3. Discontinuous Galerkin Method

In this section, we discretize the system (2.5) by a discontinuous Galerkin (DG) method [20]. Let $T = \{T_h\}_h$ be a family of conforming quasi-uniform triangulations such that $\overline{\Omega} = \cup_{\tau \in T_h} \overline{\tau}$, $\tau_i \cap \tau_j = \emptyset$ for $\tau_i, \tau_j \in T_h$, $i \neq j$. Set $h_\tau = \mathrm{diam}(\tau)$ and $h = \max_{\tau \in T_h} h_\tau$. For an integral k and $\tau \in T_h$, let $\mathbb{P}^k(\tau)$ be the set of all polynomials on τ of degree at most k. We define the discrete space

$$V_h := \{v \in L^2(\Omega) : v|_\tau \in \mathbb{P}^k(\tau) \quad \forall\, \tau \in T_h\}.$$

We split the set of all edges $\mathcal{E}_h$ into the set $\mathcal{E}_h^i$ of interior edges of T_h and the set $\mathcal{E}_h^\partial$ of boundary edges of T_h such that $\mathcal{E}_h = \mathcal{E}_h^i \cup \mathcal{E}_h^\partial$. For an $e \in \mathcal{E}_h^i$, we define the averages and jumps of $v \in V_h$ by

$$[v]_e = \lim_{\rho \to 0^+} \left[v(x - \rho n_e) - v(x + \rho n_e) \right],$$

and

$$\{v\}_e = \frac{1}{2} \lim_{\rho \to 0^+} \left[v(x - \rho n_e) + v(x + \rho n_e) \right],$$

respectively, where n_e is one of the normal unit vectors to e. For $e \in \mathcal{E}_h^\partial$, n_e denotes the outward unit normal.

We consider the discrete problem:

$$\text{Find } \gamma_{\varepsilon,h} \in V_h \text{ such that } a_h(\gamma^\delta_{\varepsilon,h}, w_h) = \sum_{e\in\mathcal{E}_h^\partial} \int_e (\beta\cdot n_e)^\ominus f' w_h\, ds \ \text{ for all } \ w_h \in V_h, \tag{3.1}$$

where the the upwind DG bilinear form a_h is given by

$$\begin{aligned} a_h(\gamma^\delta_{\varepsilon,h}, w_h) := \sum_{\tau\in T_h} \int_\tau \left[\mu_\varepsilon \gamma^\delta_{\varepsilon,h} w_h + (\beta\cdot\gamma^\delta_{\varepsilon,h}) w_h\right]\, dx + \sum_{e\in\mathcal{E}_h^\partial} \int_e (\beta\cdot n_e)^\ominus \gamma^\delta_{\varepsilon,h} w_h\, ds \\ - \sum_{e\in\mathcal{E}_h^i} \int_e (\beta\cdot n_e)[\gamma^\delta_{\varepsilon,h}]\{w_h\}\, ds + \sum_{e\in\mathcal{E}_h^i} \int_e \eta|\beta\cdot n_e|[\gamma^\delta_{\varepsilon,h}][w_h]\, ds. \end{aligned}$$

In the following, we assume that $\varepsilon > 0$ is sufficiently small. We first examine the consistency and discrete coercivity of the upwind DG bilinear form a_h. Assume that there is a partition P_Ω of Ω into disjoint polyhedra such that $\gamma^\delta_\varepsilon \in V_* := V \cap H^1(P_\Omega)$. We set $V_{*h} = V_* + V_h$. This assumption implies that $\gamma^\delta_\varepsilon \in L^2(e)$ for all $e \in \mathcal{E}_h$. The space $H^1(P_\Omega)$ can be replaced by $H^{1/2+\epsilon}(P_\Omega)$ or $W^{1,1}(P_\Omega)$ for a weakly regularity assumption where $0 < \epsilon < 1/2$. From Lemma 2.14 in [20], we know that for $u \in V_*$ and all $e \in \mathcal{E}_h^i$, $(\beta\cdot n_e)[u] = 0$ a.e. on e. Define $\tau_c = \max\{\|\mu_\varepsilon\|_{L^\infty(\Omega)}, L_\beta\}$, $\beta_c = \|\beta\|_{L^\infty(\Omega)^2}$ and the following strong norms

$$|||v|||^2 := \tau_c^{-1} \sum_{\tau\in T_h} \|v\|_{L^2(\tau)} + \frac{1}{2} \sum_{e\in\mathcal{E}_h^\partial} \int_e |\beta\cdot n_e| v^2\, ds + \sum_{e\in\mathcal{E}_h^i} \int_e \eta|\beta\cdot n_e|[v]^2\, ds,$$

and

$$|||v|||_*^2 := |||v|||^2 + \beta_c \sum_{\tau\in T_h} \|v\|^2_{L^2(\partial\tau)}$$

for $v \in V_{*h}$. For the consistency of a_h we refer to Lemma 2.27(i) in [20] and we conclude the coercivity in the following lemma.

Lemma 3.1. *For all $w_h \in V_h$, there exists a constant C_c independent of h, ε, w_h such that*

$$a_h(w_h, w_h) \ge C_c\varepsilon |||w_h|||^2. \tag{3.2}$$

Proof. It follows from the coercivity of the bilinear form a and the fact that $(\beta\cdot n_e)[w_h]$ vanishes across interior interfaces that

$$a_h(w_h, w_h) \ge \min\{1, c\tau_c\varepsilon/4\} |||w_h|||^2,$$

which yields the desired result since ε is sufficiently small. □

The discrete coercivity of a_h on V_h implies the well-posedness of the discrete problem (3.1). Let π_h be the L^2-orthogonal projection operator onto V_h. Then we have (see Theorem 2.30 in [20])

Lemma 3.2. *For $(v, w_h) \in V_* \times V_h$, there exists a constants $C_b > 0$ independent of h, ε, v, w_h such that*

$$|a_h(v - \pi_h v, w_h)| \le C_b |||v - \pi_h v|||_* |||w_h|||. \tag{3.3}$$

Now we state the main result of error estimates.

Theorem 3.1. *Let $\gamma_\varepsilon^\delta, \gamma_{\varepsilon,h}^\delta$ be the unique solution of* (2.5) *and* (3.1)*, respectively. Assume that $\gamma_\varepsilon \in H^{k+1}(\Omega)$. Then we have*

$$|||\gamma_\varepsilon^\delta - \gamma_{\varepsilon,h}^\delta||| \leq C\varepsilon^{-1}h^{k+1/2}\|\gamma_\varepsilon^\delta\|_{H^{k+1}(\Omega)}. \tag{3.4}$$

Moreover, there exists a constant $0 < s < 1/2$ such that

$$\int_\Omega |\gamma^\delta - \gamma_{\varepsilon,h}^\delta|^{1/2}\, dx \leq C\left(\varepsilon^{-1}h^{k+1/2}\|\gamma_\varepsilon\|_{H^{k+1}(\Omega)} + (1+\frac{\delta}{\varepsilon})(\varepsilon+\delta)^s\right), \tag{3.5}$$

Here, $C > 0$ is a constant independent of $\gamma^\delta, \gamma_\varepsilon^\delta, h, \varepsilon, \delta$.

Proof. Following Theorem 2.31 in [20] we know

$$|||\gamma_\varepsilon^\delta - \gamma_{\varepsilon,h}^\delta||| \leq (1 + C_1\varepsilon^{-1})|||\gamma_\varepsilon^\delta - \gamma_{\varepsilon,h}^\delta|||_* \leq C\varepsilon^{-1}|||\gamma_\varepsilon^\delta - \gamma_{\varepsilon,h}^\delta|||_*.$$

Using Lemma 1.58 and 1.59 in [20] we obtain that

$$|||\gamma_\varepsilon^\delta - \gamma_{\varepsilon,h}^\delta|||_* \leq Ch^{k+1/2}\|\gamma_\varepsilon^\delta\|_{H^{k+1}(\Omega)},$$

which implies (3.4). The estimate (3.5) follows from (2.9), (2.15), (3.4) and the triangle inequality. □

Given σ_0 and the interior data U^δ, our algorithm for the reconstruction of $\sigma|_\Omega$ is summarized as follows:

Step 1. Choose an appropriate $\varepsilon \in (0,1)$;

Step 2. Generate the data $\beta = \nabla U^\delta$ and $\mu_\varepsilon = 0.5\Delta U^\delta + \varepsilon$ (see Example 2 in Section 4);

Step 3. Solve the variational problem (12) using DG method and obtain the numerical solution $\gamma_{\varepsilon,h}^\delta$, i.e., the reconstruction of $\gamma = \sqrt{\sigma}$ in Ω.

4. Numerical Examples

In this section, we present several numerical examples for the reconstruction of σ that demonstrate the accuracy and efficiency of the proposed inversion algorithm. Although we assume Ω being a C^6-smooth bounded domain in theoretical analysis, here Ω is set to be $[0,1]\times[0,1]$ for simplicity. Since only ∇u_σ is used in the inversion we do not impose the condition $\int_\Omega u_\sigma dx = 0$ in the rest of this section. All of the numerical tests were obtained by means of Matlab numerical implementations. We always choose the parameter $\eta = 100$ used in discontinuous Galerkin method described in Section 3. The numerical errors Error and the relative L^2-error RError are calculated in accordance with the expressions

$$\text{Error} = \int_\Omega |\gamma - \gamma_{\varepsilon,h}|^{1/2}\, dx, \quad \text{and} \quad \text{RError} = \frac{\|\gamma - \gamma_{\varepsilon,h}\|_{L^2(\Omega)}}{\|\gamma\|_{L^2(\Omega)}},$$

respectively.

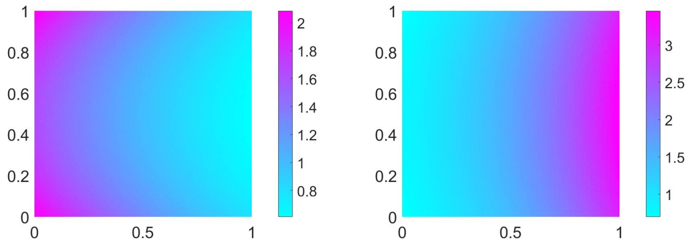
Example 4.1. Let the exact u_σ in Ω be given by

$$u_\sigma = e^{0.5 - x_1 + (x_2 - 0.5)^2},$$

and the exact conductivity is

$$\sigma = e^{3x_1 - 0.5 - (x_2 - 0.5)^2},$$

see Fig. 4.1. The triangular partition of Ω is fixed with meshsize $h = 0.0295$. The numerical errors for different k and ε are presented in Fig. 4.2. For fixed small ε, the numerical errors, dominated by the errors arising from discontinuous Galerkin approximation, decay as k increase. On the other hand, for fixed large k, the numerical errors are dominated by the term ε^s with $s = \frac{\alpha}{2(\alpha+2)}$ given in (2.14). The results demonstrate the convergence of numerical errors with respect to ε with order nearly $O(\varepsilon^{1/2})$ which corresponds to the exact theoretical rate since $|\nabla u_\sigma|$ does not vanish here (Theorem 2.2). The numerical reconstructions for $k = 3$ are shown in Fig. 4.3 with the corresponding relative L^2-errors for different choices of ε.



(a) u_σ (b) γ

Fig. 4.1. Example 1. The exact u_σ (a) and γ (b).

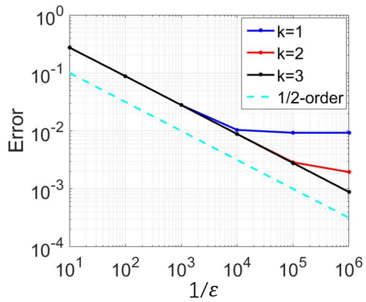
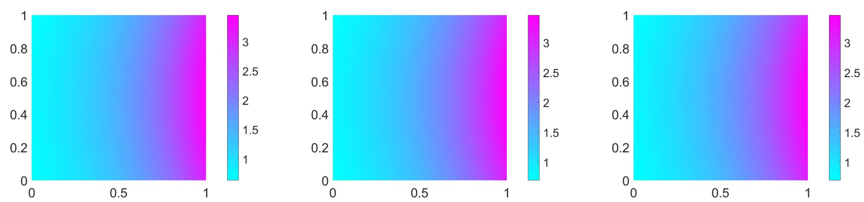


Fig. 4.2. Example 1. Numerical errors of the reconstruction for different k and ε.



(a) $\varepsilon = 10^{-1}$ (b) $\varepsilon = 10^{-3}$ (c) $\varepsilon = 10^{-5}$

Fig. 4.3. Example 1. The reconstruction of γ for different ε with relative errors RError=4.31×10^{-2}, 4.45×10^{-4} and 4.46×10^{-6} in (a,b,c), respectively.

Example 4.2. In this example, we consider the reconstruction of the conductivity function given by

$$\sigma(x_1, x_2) = q(3(2x_1 - 1), 3(2x_2 - 1)),$$

where

$$q(x_1, x_2) = 1 + 0.3(1 - x_1)^2 e^{-x_1^2-(x_2+1)^2} - (\frac{1}{5}x_1 - x_1^3 - x_2^5)e^{-x_1^2-x_2^2} - \frac{1}{30}e^{-(x_1+1)^2-x_2^2},$$

see Fig. 4.4. Set $g = e^{x_1+x_2} - (e^2 - 1)/2$ on Γ satisfying $\int_\Gamma g\,dx = 0$. The solution u_σ and $|\nabla u_\sigma|$ (produced by discontinuous Galerkin method) are presented in Fig. 4.5. For simplicity, let the exact measurement data be the polynomial coefficients of the discontinuous Galerkin approximation of u_σ in $P^{k_0}(T_h), k_0 \geq 2$. In other words, in each triangular element, we have at least $(k_0+1)(k_0+2)/2$ measurement points to produce the measurement data. The triangular partition of Ω is fixed with meshsize $h = 0.0295$. It is known that on each triangular element τ, the u_σ can be approximated by

$$u_\sigma = \sum_{j=1}^{N} u_j \phi_j,$$

with $\phi_j \in \mathbb{P}^k(\tau)$ being the basis function and u_j being the point measurements. The the terms $\beta = \nabla u_\sigma$ and $\mu_\varepsilon = 0.5\Delta u_\sigma + \varepsilon$ are calculated through

$$\beta = \sum_{j=1}^{N} u_j \nabla\phi_j, \quad \mu_\varepsilon = 0.5\sum_{j=1}^{N} u_j \Delta\phi_j + \varepsilon.$$

Choosing $k_0 = 1$ will result into poor reconstruction since in this case $\Delta u_\sigma = 0$ in each element. We choose the parameter $k = k_0$. The reconstruction results are presented in Figs. 4.6 and 4.7 and the corresponding numerical errors are displayed in Fig. 4.8. The accuracy limitation at a level of approximately 10^{-2} corresponds to the number of measurement points and the approximations of ∇u_σ and Δu_σ.

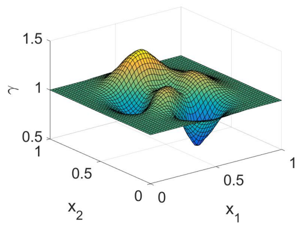
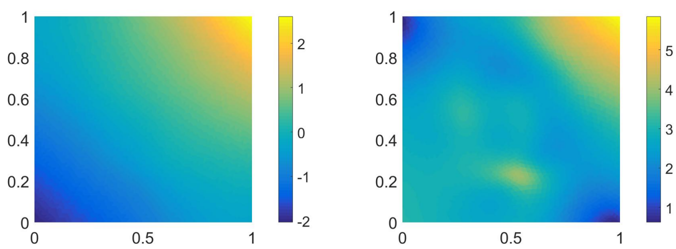
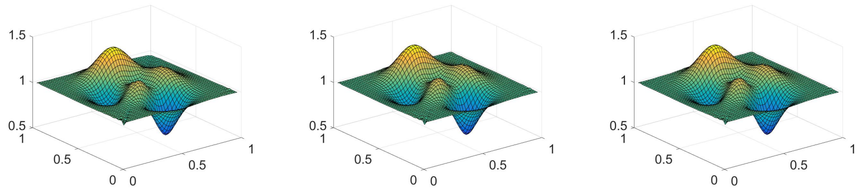


Fig. 4.4. Example 2. The exact γ.



(a) u_σ (c) $|\nabla u_\sigma|$

Fig. 4.5. Example 2. Exact data u_σ and $|\nabla u_\sigma|$.



(a) $\varepsilon = 10^{-1}$ (b) $\varepsilon = 10^{-3}$ (c) $\varepsilon = 10^{-5}$

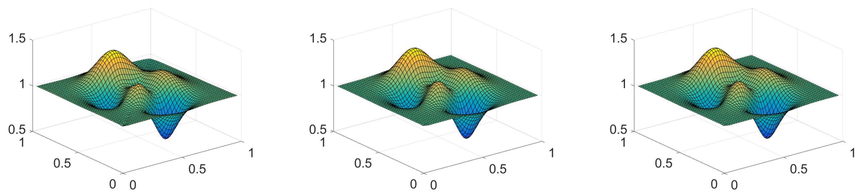
Fig. 4.6. Example 2. The reconstruction of γ when $k_0 = 2$ for different ε with relative errors RError=1.71×10^{-2}, 2.53×10^{-3} and 2.50×10^{-3} in (a,b,c), respectively.

Example 4.3. Next, we consider the reconstruction of conductivity function discussed in Example 2 from noised data

$$U^\delta = u_\sigma(1 + \delta\xi),$$

where δ is the noise level and ξ is an independent and uniformly distributed random variable generated between -1 and 1. Fig. 4.9 displays the perturbated data U^δ and ∇U^δ with $\delta = 10\%$ random noise. In addition, less point measurements are taken under a triangular partition of Ω with meshsize $h = 0.0589$ and we choose $k_0 = 2$. The reconstruction results from noised measurement with level $\delta = 5\%$ are shown in Figs. 4.10 and 4.11 which demonstrate the high efficiency and robustness of the proposed inversion algorithm. Fig. 4.12 displays the convergence of numerical errors with respect to $\delta + \varepsilon$.

Example 4.4. Finally, we consider the reconstruction of a piecewise constant conductivity, see



(a) $\varepsilon = 10^{-1}$ (b) $\varepsilon = 10^{-3}$ (c) $\varepsilon = 10^{-5}$

Fig. 4.7. Example 2. The reconstruction of γ when $k_0 = 3$ for different ε with relative errors RError=1.66×10^{-2}, 4.04×10^{-4} and 3.64×10^{-4} in (a,b,c), respectively.

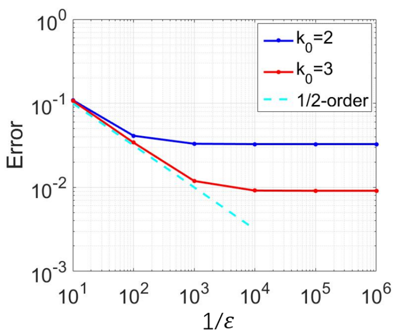
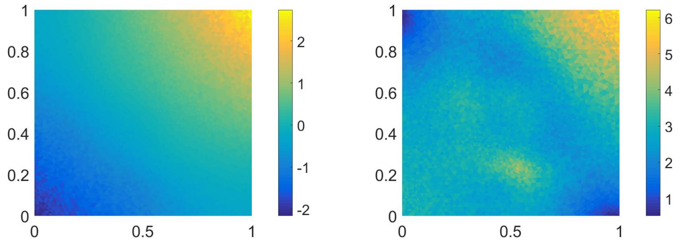
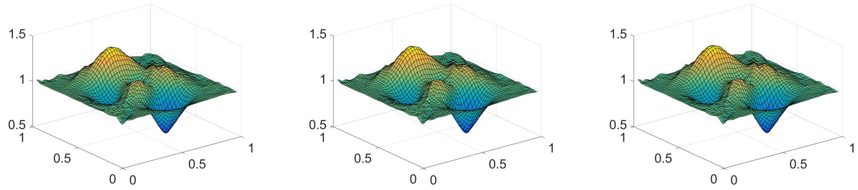


Fig. 4.8. Example 2. Numerical errors of the reconstruction for different k_0 and ε.



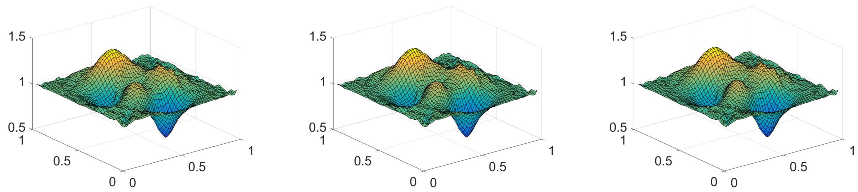
(a) U^δ (c) $|\nabla U^\delta|$

Fig. 4.9. Example 3. Perturbated data data U^δ and $|\nabla U^\delta|$ with 10% random noise.



(a) $\varepsilon = 0.1$ (b) $\varepsilon = 0.06$ (c) $\varepsilon = 0.01$

Fig. 4.10. Example 3. The reconstruction of γ from noised measurements when $k_0 = 2$, $\delta = 5\%$ for different ε with relative errors RError=2.24×10^{-2}, 1.71×10^{-2} and 1.22×10^{-2} in (a,b,c), respectively.



(a) $\varepsilon = 0.1$ (b) $\varepsilon = 0.06$ (c) $\varepsilon = 0.01$

Fig. 4.11. Example 3. The reconstruction of γ from noised measurements when $k_0 = 2$, $\delta = 10\%$ for different ε with relative errors RError=2.46×10^{-2}, 2.04×10^{-2} and 1.74×10^{-2} in (a,b,c), respectively.

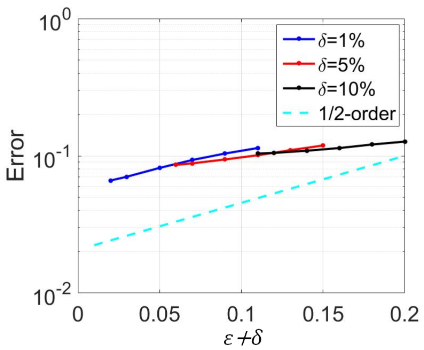


Fig. 4.12. Example 3. Numerical errors of the reconstruction for different noise level δ and ε.

Fig. 4.13. For simplicity, the exact u_σ in Ω is set to be

$$u_\sigma = \cos(x_1 - 0.5)e^{x_2}.$$

The reconstruction from perturbated point measurement with 10% random noise presented in Fig. 4.14 shows the efficiency of the proposed method.

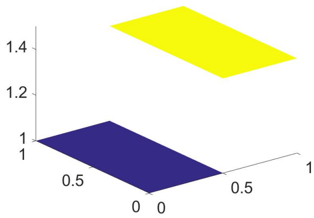
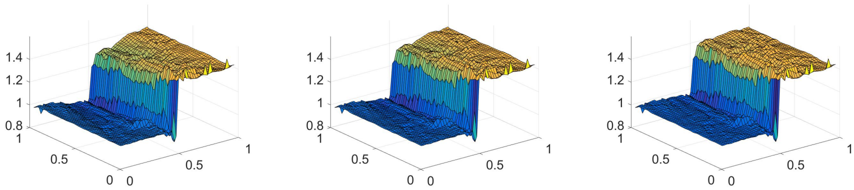


Fig. 4.13. Example 4. The exact γ.



(a) $\varepsilon = 0.1$ (b) $\varepsilon = 0.06$ (c) $\varepsilon = 0.01$

Fig. 4.14. Example 4. The reconstruction of γ from noised measurements when $k_0 = 2$, $\delta = 10\%$ for different ε with relative errors RError=3.79×10^{-2}, 2.88×10^{-2} and 2.29×10^{-2} in (a,b,c), respectively.

Appendix A

Lemma .1 (Traces and integration by parts [18, 20]). *The trace operator*

$$\gamma : C^0(\overline{\Omega}) \ni v \mapsto \gamma(v) := v|_{\partial\Omega} \in L^2(|\beta \cdot \nu|, \partial\Omega)$$

extends continuously to V, meaning that there is C_γ such that, for all $v \in V$,

$$\|\gamma(v)\|_{L^2(|\beta \cdot n|, \partial\Omega)} \leq C_\gamma \|v\|_V.$$

Moreover, the following integration by parts formula holds true: For all $v, w \in V$,

$$\int_\Omega [(\beta \cdot \nabla v)w + (\beta \cdot \nabla w)v + (\nabla \cdot \beta)vw]\, dx = \int_{\partial\Omega} (\beta \cdot \nu)\gamma(v)\gamma(w) ds.$$

Theorem .1 (The Banach-Nečas-Babuška Theorem [23]). *Let X be a Banach space and let Y be a reflexive Banach space. Let $a \in \mathcal{L}(X \times Y, \mathbb{R})$ and let $f \in Y'$. Then the problem:*

$$\textit{Find } u \in X \textit{ such that } a(u, w) = \langle f, w \rangle_{Y',Y} \textit{ for all } w \in Y.$$

is well-posed if and only if:
(i). *There is $C_{sta} > 0$ such that*

$$C_{sta}\|v\|_X \leq \sup_{0 \neq w \in Y} \frac{a(v, w)}{\|w\|_Y}, \qquad \forall v \in X. \tag{.1}$$

(ii). *For all $w \in X$,*

$$(\forall v \in X, a(v, w) = 0) \implies (v = 0). \tag{.2}$$

Proof. [Lemma 2.1] Further $c > 0$ is a constant that only depends on g, σ, Ω, and eventually on η. Since u_σ is a solution to the system (1.1), $|\nabla u_\sigma|$ is a Mukenhoupt weight. Indeed there exists a contant $p > 1$ depending only on g, σ and Ω such that the following inequality (Theorem 1.1 in [24])

$$\left(\frac{1}{|B_r(x)|}\int_{B_r(x)} |\nabla u_\sigma| dy\right)\left(\frac{1}{|B_r(x)|}\int_{B_r(x)} |\nabla u_\sigma|^{-\frac{1}{p-1}} dy\right)^{p-1} \leq c, \tag{.3}$$

holds for all $r \in (0, \eta)$, and $x \in \Omega_\eta$.

The following behavior is related to the unique continuation properties of solutions to elliptic equations in a divergence form (Corollary 3.1 in [3]).

$$cr^\beta \leq \frac{1}{|B_r(x)|}\int_{B_r} |\nabla u_\sigma| dy, \tag{.4}$$

where $\beta \geq 0$ is a constant that only depends on g, σ and Ω. The constants β and $p > 1$ are related to the vanishing order of ∇u_σ in Ω_η. Combining inequalities (.3) and (.4), we obtain

$$\int_{B_r(x)} |\nabla u_\sigma|^{-\frac{1}{p-1}} dy \leq cr^{\frac{-\beta+n}{p-1}}, \tag{.5}$$

for all $r \in (0, \eta)$, and $x \in \Omega_\eta$.

Fix now $r = \frac{\eta}{2}$. There exist $N \in \mathbb{N}$ and x_j, $1 \leq j \leq N$, that only depend on η and Ω such that $\Omega_\eta \subset \cup_{j=1}^N B_r(x_j)$. Let $(\phi_j)_{1\leq j\leq N}$ be a partition of unity subordinate to the covering $\cup_{j=1}^N B_r(x_j)$. We deduce from (.5), the following estimates

$$\begin{aligned}\int_{\Omega_\eta} |\nabla u_\sigma|^{-\frac{1}{p-1}} dy &= \sum_{j=1}^N \int_{\Omega_\eta} |\nabla u_\sigma|^{-\frac{1}{p-1}} \phi_j dy\\ &\leq \sum_{j=1}^N \|\phi_j\|_{L^\infty(\Omega_\eta)} \int_{B_r(x_j)} |\nabla u_\sigma|^{-\frac{1}{p-1}} dy\\ &\leq c.\end{aligned} \tag{.6}$$

Recall $\Omega_\eta \setminus \overline{\Omega_\eta^t} = \{x \in \Omega_\eta : |\nabla u_\sigma| < t\}$. Assuming that $\Omega_\eta \setminus \overline{\Omega_\eta^t}$ is not empty, we infer from (.6) the following inequality

$$t^{-\frac{1}{p-1}} |\Omega_\eta \setminus \overline{\Omega_\eta^t}| \leq \int_{\Omega_\eta\setminus\overline{\Omega_\eta^t}} |\nabla u_\sigma|^{-\frac{1}{p-1}} dy \leq \int_{\Omega_\eta} |\nabla u_\sigma|^{-\frac{1}{p-1}} dy \leq c,$$

which in turn leads to

$$0 \leq |\Omega_\eta \setminus \overline{\Omega_\eta^t}| \leq ct^\alpha,$$

with $\alpha := \frac{1}{p-1} > 0$. ☐

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