J. Math. Study
doi: 10.4208/jms.vxxnx.xx.xx

Vol. **xx**, No. x, pp. 1-20
xxxxxx 20xx

# Anisotropic Elliptic Nonlinear Obstacle Problem with Weighted Variable Exponent

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Received December 2, 2019; Accepted March 24, 2020;
Published online June 29, 2020.

**Abstract.** In this paper, we are concerned with a show the existence of a entropy solution to the obstacle problem associated with the equation of the type :

$$\begin{cases} Au+g(x,u,\nabla u)=f & \text{in} \quad \Omega, \\ u=0 & \text{on} \quad \partial\Omega, \end{cases}$$

where $\Omega$ is a bounded open subset of $\mathbb{R}^N$, $N\geq 2$, $A$ is an operator of Leray-Lions type acting from $W_0^{1,\overrightarrow{p}(.)}(\Omega,\ \overrightarrow{w}(.))$ into its dual $W_0^{-1,\overrightarrow{p}'(.)}(\Omega,\ \overrightarrow{w}^*(.))$ and $L^1-$deta. The nonlinear term $g$: $\Omega\times\mathbb{R}\times\mathbb{R}^N\longrightarrow\mathbb{R}$ satisfying only some growth condition.

**AMS subject classifications**: 35J60, 35J87, 35J66.

**Key words**: Entropy solutions, Anisotropic elliptic equations, weighted anisotropic variable exponent Sobolev space.

## 1 Introduction

Consider $\Omega$ be a bounded open subset of $\mathbb{R}^N(N\geq 2)$ and $p_i(.)\in C_+(\overline{\Omega})$ for $i=0,1,\ldots,N$, with for all $x$ in $\Omega$,

$$p_0(x)\geq \max\{p_i(x),\ i=1,\ \ldots,\ N\}. \tag{1.1}$$

Let $W^{1,\overrightarrow{p}(.)}(\Omega,\overrightarrow{w}(.))$ be weighted anisotropic variable exponent Sobolev space associated to the vector $\overrightarrow{p}(.)$, with $\overrightarrow{p}(.)=\{p_0(\cdot),\ldots,p_N(.)\}$, where $p_0(x),p_1(x),\ldots,p_N(x)$ be $N+1$ variable exponents and $\overrightarrow{w}(.)$ denoting a vector of measurable positive functions, i.e., $\overrightarrow{w}(.)=\{w_1(\cdot),\ldots,w_N(.)\}$, with $w_i$ are weight measurable functions for all $i=1,\cdots,N$.

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Our aim is to prove the existence of solutions with respect to perturbations in the growth exponent p of the following problems:

$$\begin{cases} -\sum_{i=1}^{N} D^i a_i(x,u,\nabla u) + g(x,u,\nabla u) = f & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{$\mathcal{P}$}$$

in the convex class $K_\psi := \left\{ u \in W_0^{1,\overrightarrow{p}(x)}(\Omega, \overrightarrow{w}(x)), u \geq \psi \quad \text{a.e in } \Omega \right\}$, where $\psi$ is a fixed obstacle function, such that

$$\psi^+ \in W_0^{1,\overrightarrow{p}(x)}(\Omega, \overrightarrow{w}(x)) \cap L^\infty(\Omega). \tag{1.2}$$

We assume that $a_i : \Omega \times \mathbb{R} \times \mathbb{R}^N \mapsto \mathbb{R}$ are Carathodory functions for $i = 1,2,\ldots,N$, (measurable with respect to $x$ in $\Omega$ for every $(s, \xi)$ in $\mathbb{R} \times \mathbb{R}^N$ and continuous with respect to $(s, \xi)$ in $\mathbb{R} \times \mathbb{R}^N$ for almost every $x$ in $\Omega$) which satisfies the following conditions:

$$a_i(x, s, \xi)\xi_i \geq \alpha w_i \left|\xi_j\right|^{p_i(x)} \qquad \text{for} \quad i = 1,\ldots,N, \tag{1.3}$$

$$|a_i(x, s, \xi)| \leq \beta w_i^{\frac{1}{p_i(x)}} \left( M_i(x) + |s|^{p_i(x)-1} + w_i^{\frac{1}{p_i'(x)}} |\xi_i|^{p_i(x)-1} \right) \quad \text{for} \quad i = 1,\ldots,N, \tag{1.4}$$

for all $\xi = (\xi_1, \ldots, \xi_N)$ and $\xi' = (\xi_1', \ldots, \xi_N')$ , we have

$$(a_i(x, s, \xi) - a_i(x, s, \xi'))(\xi_i - \xi_i') > 0 \quad \text{for} \quad \xi_i \neq \xi_i', \tag{1.5}$$

for a.e. $x \in \Omega$, and all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$, where $M_i(.)$ is a nonnegative function lying in $L^{p_i'(.)}(\Omega)$ and $\alpha, \beta > 0$.

The nonlinear term $g(x, s, \xi)$ is a Caratheodory function which satisfies only the growth condition

$$|g(x, s, \xi)| \leq c(x) + b(|s|) \sum_{i=1}^{N} w_i |\xi_i|^{p_i(x)} \tag{1.6}$$

where $b : \mathbb{R} \mapsto \mathbb{R}^+$ is a continuous positive function that belongs to $L^1(\mathbb{R})$ and $c(x) \in L^1(\Omega)$.

In the particular case when $p_i = p$ for any $i \in \{1,...,N\}$, Yazough, Azroul and Redwane (see [16]) have proved the existence of entropy solutions to problem like $(\mathcal{P})$. Then, Azroul, Benboubker and Ouaro [6] have obtained the above results via penalization methods.

The study of $(\mathcal{P})$ is a new and interesting topic when the data is in $L^1$. One result on this topic can be found in [5,8,11], where the discussion was conducted in the framework of weighted anisotropic Sobolev space with variable exponent (we refer to [1, 2, 11] for more details), the notion of a entropy solution was introduced by Benilan et. al [7,9] and P.-L. Lions [14] in their study of the Boltzmann equation. We mention some works in the direction of the anisotropic space such as [4,8].

The aim of this paper is to extend the results in [5] to the anisotropic obstacle non-linear elliptic problem. We want to prove only existence results, the uniqueness problem being a rather delicate one, this kind of problems still attracting the interest of the researchers (see [10, 11, 15] for a survey). One of the motivations for studying $(\mathcal{P})$ comes from applications to elasticity as the equations that models the shape of an elastic membrane which is pushed by an obstacle from one side affecting its shape. The layout of the paper is presented as follows : Section 2 contains a brief discussion of variable exponent Lebesgue with weighted and the weighted anisotropic variable exponent Sobolev space, in Section 3 we introduce some useful technical lemmas, we prove our main result in Section 4.

## 2 Preliminaries

In this section, we state some elementary properties of the weighted variable exponent Lebesgue-Sobolev spaces which will be used in the next sections. The basic properties of the variable exponent Lebesgue-Sobolev spaces. Let $\Omega$ be a bounded open subset of $\mathbb{R}^N (N \geq 2)$ , we assume that the variable exponent $p(\cdot): \overline{\Omega} \to [1,+\infty[$ is log-Hölder continuous on $\Omega$, that is there is a real constant $c>0$ such that for all $x,y \in \overline{\Omega}$, $x \neq y$ with $|x-y| < \frac{1}{2}$ one has:

$$|p(x)-p(y)| \leq \frac{c}{-\log|x-y|}$$

and satisfying

$$p^- \leq p(x) \leq p^+ < +\infty,$$

where

$$p^- := \operatorname{ess}\inf_{x\in\overline{\Omega}} p(x), \qquad p^+ := \operatorname{ess}\sup_{x\in\overline{\Omega}} p(x).$$

For almost everywhere strictly positive and measurable function $w: \Omega \to \mathbb{R}$ will be called a weight. We shall denote by $L^{p(\cdot)}(\Omega, w)$ the set of all measurable functions $u$ on $\Omega$ such that the norm

$$\|u\|_{p(x),w(x)} = \inf\left\{ \mu > 0 : \int_\Omega w(x) \left|\frac{u}{\mu}\right|^{p(x)} dx \leq 1 \right\},$$

is finite. $L^{p(\cdot)}(\Omega, w)$ is also called weighted Lebesgue space.

**Proposition 2.1.** ([1]) the space $\left(L^{p(x)}(\Omega,w), \|.\|_{p(x),w}\right)$ is of Banach.

**Remark 2.1.** In simple case $w(x)=1$, we find again the Lebesgue space with variable exponents $L^{p(x)}(\Omega)$; and $\rho_w(u) = \rho_1(u) := \rho(u) = \int_\Omega |u|^{p(x)} dx$, (see [12, 17]).

**Lemma 2.1.** *For all function $u \in L^{p(x)}(\Omega, w)$. There are the following assertions:*

1. *If* $\rho_{w(x)}(u)>1\ (=1;<1)\Leftrightarrow\|u\|_{p(x),w(x)}>1\ (=1;<1)$, *respectively.*

2. *If* $\|u\|_{p(x),w(x)}>1$, *then* $\|u\|^{p_-}_{p(x),w(x)}\le\rho_{w(x)}(u)\le\|u\|^{p^+}_{p(x),w(x)}$.

3. *If* $\|u\|_{p(x),w(x)}<1$, *then* $\|u\|^{p^+}_{p(x),w(x)}\le\rho_{w(x)}(u)\le\|u\|^{p_-}_{p(x),w(x)}$.

*Proof.* Seeing that $\rho_w(u)=\rho(w^{\frac{1}{p(x)}}u)$ and $\|w^{\frac{1}{p(x)}}u\|_{p(x)}=\|u\|_{p(x),w}$, and using [17], we prove Lemma 2.1 above. □

Throughout the paper, we assume that $w_i$ a weight function for any $i=1,\dots,N$, satisfying the conditions:

$$w_i\in L^1_{loc}(\Omega),\qquad w_i^{\frac{-1}{p_i(x)-1}}\in L^1_{loc}(\Omega), \tag{$H_1$}$$

$$\omega_i^{-s(x)}\in L^1(\Omega)\quad\text{with}\quad s(x)\in\left(\frac{N}{p_i(x)},\infty\right)\cap\left[\frac{1}{p_i(x)-1},\infty\right). \tag{$H_2$}$$

The reasons why we assume **(**$H_1$**)** and **(**$H_2$**)** can be found in [13].

**Proposition 2.2.** Let $\Omega$ be a bounded open subset of $\mathbb{R}^N$, and $w_i$ be a weight function on $\Omega$, for any $i=1,\dots,N$. If **(**$H_1$**)** is verified then for all $i=1,\dots,N$ we have $L^{p_i(x)}(\Omega,w_i)\hookrightarrow L^1_{loc}(\Omega)$.

*Proof.* Let $K$ be a included compact on $\Omega$. Using Hölder inequality for all $i=1,\dots,N$ we have

$$\begin{aligned}\int_K|u|dx&=\int_K|u|w_i^{\frac{1}{p_i(x)}}w_i^{\frac{-1}{p_i(x)}}dx\\&\le2\||u|w_i^{\frac{1}{p_i(x)}}\|_{L^{p_i(x)}(K)}\|w_i^{\frac{-1}{p_i(x)}}\|_{L^{p_i'(x)}(K)},\\&\le2\|u\|_{p_i(x),w_i(x)}\left(\int_K w_i^{\frac{-p_i'(x)}{p_i(x)}}dx+1\right)^{\frac{1}{p'_{i-}}},\\&\le2\|u\|_{p_i(x),w_i(x)}\left(\int_K w_i^{\frac{-1}{p_i(x)-1}}dx+1\right)^{\frac{1}{p'_{i-}}}.\end{aligned}$$

Thanks to the assumption **(**$H_1$**)** we deduce that $\int_K|u|dx\le C\|u\|_{p(x),w_i(x)}$. □

Now, we present the weighted anisotropic variable exponent Sobolev space, used in the study of the unilateral elliptic problem $(\mathcal{P})$. Let $p_0(x),p_1(x),\dots,p_N(x)$ be $N+1$ variable exponents in $C_+(\Omega)$ and $w_i$ are weight measurable functions for all $i=1,\cdots,N$. We define the following vectors

$$\overrightarrow{p}(.)=\{p_0(\cdot),\dots,p_N(.)\}\quad\text{and}\quad\overrightarrow{w}(.)=\{w_1(\cdot),\dots,w_N(.)\}.$$

We denote

$$D^0 u = u \quad \text{and} \quad D^i u = \frac{\partial u}{\partial x_i} \quad \text{for} \quad i = 1,\ldots,N,$$

and

$$\underline{p} = \min\{p_0^-, p_1^-, \ldots, p_N^-\} \quad \text{then} \quad \underline{p} > 1. \tag{2.1}$$

The weighted anisotropic variable exponent Sobolev space $W^{1,\overrightarrow{p}(.)}(\Omega,\overrightarrow{w}(.))$ is defined as the collection of all functions $u \in L^{p_0(.)}(\Omega)$, and the derivatives $D^i u$ are in $L^{p_i(x)}(\Omega, w_i)$ for all $i = 1,\ldots,N$ is a Banach space with respect to norm (see [11])

$$\|u\|_{1,\overrightarrow{p}(.),\overrightarrow{w}(.)} = \sum_{i=0}^{N} \|D^i u\|_{p_i(.),w_i(.)}. \tag{2.2}$$

We denote by $C_0^\infty(\Omega)$ the space of all functions with compact support in $\Omega$ with continuous derivatives of arbitrary order. We define the functional space $W_0^{1,\overrightarrow{p}(.)}(\Omega,\overrightarrow{w}(.))$ as the closure of $C_0^\infty(\Omega)$ in $W^{1,\overrightarrow{p}(.)}(\Omega,\ \overrightarrow{w}(.))$ with respect to the norm (2.2). Note that $C_0^\infty(\Omega)$ is dense in $W_0^{1,\overrightarrow{p}(.)}(\Omega,\overrightarrow{w}(.))$. By an adapted method of that of Adams [1], and by constructing an isometric isomorphism from $W^{1,\overrightarrow{p}(.)}(\Omega,\ \overrightarrow{w}(.))$ into $\prod_{i=0}^{N} L^{p_i(.)}(\Omega, w_i(.))$, we can show that $\left(W_0^{1,\overrightarrow{p}}(\Omega,\overrightarrow{w}(.)), \|.\|_{1,\overrightarrow{p}(.),\overrightarrow{w}(.)}\right)$ is separable and reflexive if $1 \le p_i(.) < \infty$ and $1 < p_i(.) < \infty$, respectively, for all $i = 1,\ldots,N$. For $p_i(.) > 1, W^{-1,\overrightarrow{p'}(.)}(\Omega,\ \overrightarrow{w^*}(.))$ designs its dual where $\overrightarrow{p'}(.)$ is the conjugate of $\overrightarrow{p}(.)$, i.e., $p_i'(.) = \dfrac{p_i(.)}{p_i(.)-1}$ and

$$\overrightarrow{w^*}(.) = \left\{ w_i^*(.) = w_i^{1-p_i'(.)}(.), i = 1,\ldots,N \right\}.$$

Let us introduce the function $p_s$ defined by

$$p_s(x) = \frac{\underline{p}(x)s(x)}{s(x)+1},$$

we have

$$p_s(x) < p(x) \quad \text{a.e. in} \quad \Omega,$$

and

$$\begin{cases} p_s^*(x) = \dfrac{N p_s(x)}{N - p_s(x)} & \text{if } p(x)s(x) < N(s(x)+1), \\ p_s^*(x) \text{arbitrary}, & \text{else if.} \end{cases}$$

**Lemma 2.2.** *Let $\Omega$ be a smooth bounded open subset of $\mathbb{R}^N$, and suppose that $\inf w_i(.) > 0$ a.e. in $\Omega$ for all $i = 1,\ldots,N$. Let ($H_1$) and ($H_2$) be satisfied, we have the following continuous and compact embedding*

- *If $\underline{p} < N$, then $W_0^{1,\overrightarrow{p}(.)}(\Omega,\ \overrightarrow{w}(.)) \hookrightarrow\hookrightarrow L^{q(.)}(\Omega)$ for all $q \in \left[\underline{p}, p_s^*\right[$,*

- *If* $\underline{p}=N$, *then* $W_0^{1,\overrightarrow{p}(.)}(\Omega,\ \overrightarrow{w}(.))\hookrightarrow\hookrightarrow L^{q(.)}(\Omega)$ *for all* $q\in[\underline{p},+\infty[$,
- *If* $\underline{p}>N$, *then* $W_0^{1,\overrightarrow{p}(.)}(\Omega,\ \overrightarrow{w}(.))\hookrightarrow\hookrightarrow L^{\infty}(\Omega)\cap C^0(\overline{\Omega})$.

The proof of this lemma follows from the fact that the embedding

$$W_0^{1,\overrightarrow{p}(.)}(\Omega,\ \overrightarrow{w}(.))\subset W_0^{1,\overrightarrow{p_s}(.)}(\Omega)\subset W_0^{1,\underline{p}}(\Omega)$$

is continuous, and in view of the compact embedding theorem $W_0^{1,\overrightarrow{p}(.)}(\Omega,\ \overrightarrow{w}(.))$ for Sobolev spaces. Moreover, we consider

$$\mathcal{T}_0^{1,\overrightarrow{p}(.)}(\Omega,\overrightarrow{w}(.)):=\Big\{u:\Omega\mapsto\mathbb{R},\text{measurable, such that}\,T_k(u)\in W_0^{1,\overrightarrow{p}(.)}(\Omega,\overrightarrow{w}(.)),\quad\forall k>0\Big\},$$

where

$$T_k(s)=\begin{cases} s, & \text{if} \quad |s|\leq k,\\ k\dfrac{s}{|s|}, & \text{if} \quad |s|>k.\end{cases}$$

## 3 Technical lemmas

In this section, we introduce some useful technical lemmas to show our main result.

**Lemma 3.1** ([1]). *For any* $i=1,\dots,N$, *let* $w_i(.)$ *be a function weight in* $\Omega$, $r_i(.)\in C_+(\overline{\Omega})$, $g\in L^{r_i(x)}(\Omega,w_i(.))$ *and* $(g_n)_n\subset L^{r_i(x)}(\Omega,w_i(.))$ *such that* $\|g_n\|_{r_i(x),w_i(.)}\leq C$, *if* $g_n\to g$ *a.e. in* $\Omega$, *then* $g_n\rightharpoonup g$ *weakly in* $L^{r_i(x)}(\Omega,w_i(.))$.

**Lemma 3.2** ([1]). *Assume that* (1.3) - (1.5) *hold, let* $(u_n)_n$ *a sequence in* $W_0^{1,\overrightarrow{p}(.)}(\Omega,\ \overrightarrow{w}(.))$ *and* $u\in W_0^{1,\overrightarrow{p}(.)}(\Omega,\ \overrightarrow{w}(.))$. *If*

$$u_n\rightharpoonup u\quad \text{weakly in}\quad W_0^{1,\overrightarrow{p}(.)}(\Omega,\ \overrightarrow{w}(.)),$$

*and*

$$\sum_{i=1}^{N}\int_\Omega (a_i(x,\ u_n,\ \nabla u_n)-a_i(x,\ u_n,\ \nabla u))(D^iu_n-D^iu)dx\to 0,$$

*then* $u_n\longrightarrow u$ *strongly in* $W_0^{1,\overrightarrow{p}(.)}(\Omega,\ \overrightarrow{w}(.))$.

**Lemma 3.3.** *Let* $(u_n)_n$ *a sequence from* $W_0^{1,\overrightarrow{p}(.)}(\Omega,\ \overrightarrow{w}(.))$ *such that* $u_n\rightharpoonup u$ *weakly in* $W_0^{1,\overrightarrow{p}(.)}(\Omega,\ \overrightarrow{w}(.))$. *Then* $T_k(u_n)\rightharpoonup T_k(u)$ *weakly in* $W_0^{1,\overrightarrow{p}(.)}(\Omega,\ \overrightarrow{w}(.))$.

*Proof.* We have $u_n \rightharpoonup u$ weakly in $W_0^{1,\overrightarrow{p}(.)}(\Omega,\ \overrightarrow{w}(.))$ and $W_0^{1,\overrightarrow{p}(.)}(\Omega,\ \overrightarrow{w}(.)) \hookrightarrow\hookrightarrow L^{q(x)}(\Omega)$, we will have $u_n \to u$ strongly in $L^{q(x)}(\Omega)$ and a.e. in $\Omega$, consequently $T_k(u_n) \to T_k(u_n)$ a.e. in $\Omega$.

On another side

$$\begin{aligned}\sum_{i=1}^{N} \|T_k(u_n)\|_{p_i(x),w_i(x)}^{\gamma_1} &\leq \sum_{i=1}^{N} \int_\Omega |\nabla T_k(u_n)|^{p_i(x)} w_i(x)dx, \\ \leq \ \sum_{i=1}^{N} \int_\Omega |\nabla u_n|^{p_i(x)} w_i(x)dx &\leq \sum_{i=1}^{N} |u_n|_{p_i(x),w_i(x)}^{\gamma_2}\end{aligned}$$

where

$$\gamma_1 = \begin{cases} p_i^+ & \text{if} \quad \|T_k(u_n)\|_{p_i(x),w_i(x)} \leq 1, \\ p_{i-} & \text{if} \quad \|T_k(u_n)\|_{p_i(x),w_i(x)} > 1, \end{cases}$$

$$\gamma_2 = \begin{cases} p_i^+ & \text{if} \quad \|u_n\|_{p_i(x),w_i(x)} \geq 1, \\ p_{i-} & \text{if} \quad \|u_n\|_{p_i(x),w_i(x)} < 1. \end{cases}$$

Finally, we obtain

$$\|T_k(u_n)\|_{1,\overrightarrow{p}(.),\overrightarrow{w}(.)} \leq C\|u_n\|_{1,\overrightarrow{p}(.),\overrightarrow{w}(.)},$$

or $C$ is a constant positive. Thus $(T_k(u_n))_n$ is bounded in $W_0^{1,\overrightarrow{p}(.)}(\Omega,\ \overrightarrow{w}(.))$, consequently $T_k(u_n) \rightharpoonup T_k(u)$ weakly in $W_0^{1,\overrightarrow{p}(.)}(\Omega,\ \overrightarrow{w}(.))$. □

## 4 Main results

In this section we formulate and prove the main result of the paper.

Now, we give a definition of entropy solutions for our unilateral elliptic problem $(\mathcal{P})$.

**Definition 4.1.** A measurable function $u$ is said to be an entropy solution for the obstacle problem $(\mathcal{P})$, if $u \in \mathcal{T}_0^{1,\overrightarrow{p}(.)}(\Omega,\overrightarrow{w}(.))$ such that $u \geq \psi$ a.e. in $\Omega$ and

$$\begin{aligned}&\sum_{i=1}^{N} \int_\Omega a_i(x,u,\nabla u) D^i T_k(u-\varphi)dx + \int_\Omega g(x,\ u,\ \nabla u) T_k(u-\varphi)dx \\ &\leq \int_\Omega f T_k(u-\varphi)dx, \qquad \forall \varphi \in K_\psi \cap L^\infty(\Omega). \end{aligned} \tag{4.1}$$

**Theorem 4.1.** *Assume that* (1.1)$-$(1.6) *holds and* $f \in L^1(\Omega)$. *Then the problem* $(\mathcal{P})$ *admits at least one entropy solution.*

*Proof.* **Step 1 : Approximate problem**

Let $(f_n)_n$ be a sequence of smooth functions such that $f_n \to f$ in $L^1(\Omega)$ with $|f_n| \leq |f|$.

We consider the following sequence of approximate problems

$$\begin{cases} u_n \in K_\psi, \\ \sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u_n), \nabla u_n)(D^i u_n - D^i v)dx + \int_\Omega g_n(x,\, u_n,\, \nabla u_n)(u_n - v)dx \\ \qquad \leq \int_\Omega f_n(u_n - v)dx, \quad \forall v \in K_\psi, \end{cases} \tag{$\mathcal{P}_n$}$$

where $g_n(x,\, s,\, \xi) = \dfrac{g(x,s,\xi)}{1+\frac{1}{n}|g(x,s,\xi)|}$. Note that

$$|g_n(x,\, s,\, \xi)| \leq |g(x,\, s,\, \xi)| \quad \text{and} \quad |g_n(x,\, s,\, \xi)| \leq n, \qquad \forall n \in \mathbb{N}^*.$$

Let pose $V = K_\psi$ and define the operators $A_n$ and $G_n$ of $V$ to $V^*$ by :

$$\langle A_n u, v\rangle = \sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u_n), \nabla u_n) D^i v dx \quad \text{and} \quad \langle G_n u, v\rangle = \int_\Omega g_n(x,\, u,\, \nabla u) v dx.$$

**Lemma 4.1.** *The operator* $G_n$ *is bounded.*

*Proof.* In light of Hölder's type inequality, we have for all $u, v \in V$,

$$\begin{aligned} &\left| \int_\Omega g_n(x,\, u,\, \nabla u) v dx \right| \\ &\leq \int_\Omega |g_n(x,\, u,\, \nabla u)| w_i^{\frac{1}{p_i(x)}}(x) w_i^{\frac{-1}{p_i(x)}}(x) v dx \\ &\leq \sum_{i=1}^{N} \left( \frac{1}{p_{i-}} + \frac{1}{p'_{i-}} \right) \left\| |g_n(x,u,\nabla u)| w_i(x)^{\frac{-1}{p_i(x)}} \right\|_{p'_i(x)} \| v w_i(x)^{\frac{1}{p_i(x)}} \|_{p_i(x)}, \\ &\leq \sum_{i=1}^{N} C_i \left( \int_\Omega |g_n(x,u,\nabla u)|^{p'_i(x)} w_i^*(x) dx \right)^{\frac{1}{\theta_1}} \|v\|_{1,\vec{p}(.),\vec{w}(.)} \\ &\leq C n^{\frac{p'_+}{\theta_1}} \left( \int_\Omega w_i^*(x) dx \right)^{\frac{1}{\theta_1}} \|v\|_{1,\vec{p}(.),\vec{w}(.)} \\ &\leq C' \|v\|_{1,\vec{p}(.),\vec{w}(.)}, \end{aligned} \tag{4.2}$$

where

$$\theta_1 = \begin{cases} (\frac{p_i}{p_{s_i}})_- & \text{if} \quad \left\| |v(x)|^{p_{s_i}(x)} \nu^{\frac{p_{s_i}(x)}{p_i(x)}} \right\|_{\frac{p_i(x)}{p_{s_i}(x)}} \geq 1, \\ (\frac{p_i}{p_{s_i}})^+ & \text{if} \quad \left\| |v(x)|^{p_{s_i}(x)} \nu^{\frac{p_{s_i}(x)}{p_i(x)}} \right\|_{\frac{p_i(x)}{p_{s_i}(x)}} < 1. \end{cases}$$

□

**Lemma 4.2.** *The operator* $S_n = A_n + G_n : W_0^{1,\overrightarrow{p}(.)}(\Omega, \overrightarrow{w}(.)) \longrightarrow W_0^{-1,\overrightarrow{p'}(.)}(\Omega, \overrightarrow{w^*}(.))$ *is pseudo monotone and coercive in the following sense : there exists* $v_0 \in K_\psi$ *such that*

$$\frac{\langle S_n v, v - v_0\rangle}{\|v\|_{1,\overrightarrow{p}(\cdot),\overrightarrow{w}(\cdot)}} \longrightarrow +\infty \quad if \quad \|v\|_{1,\overrightarrow{p}(\cdot),\overrightarrow{w}(\cdot)} \to +\infty, \quad for \quad v \in K_\psi.$$

*Proof.* In light of the generalized Hölder′s type inequality and the growth condition (1.4), we can prove the operator $A_n$ is bounded, and by using Lemma 4.1 we conclude that $S_n$ is bounded.

We show that $S_n$ is pseudo-monotone. Let $(u_k)_{k\in\mathbb{N}}$ be a sequence in $W_0^{1,\overrightarrow{p}(.)}(\Omega, \overrightarrow{w}(.))$, such that

$$\begin{cases} u_k \rightharpoonup u & \text{in} \quad W_0^{1,\overrightarrow{p}(.)}(\Omega, \overrightarrow{w}(.)), \\ S_n u_k \rightharpoonup \chi_n & \text{in} \quad W_0^{-1,\overrightarrow{p'}(.)}(\Omega, \overrightarrow{w^*}(.)), \\ \limsup\limits_{k\to\infty} \langle S_n u_k, u_k\rangle \le \langle \chi_n, u\rangle. \end{cases} \tag{4.3}$$

We prove that

$$\chi_n = S_n u \qquad \text{and} \qquad \langle S_n u_k, u_k\rangle \to \langle \chi_n, u\rangle \qquad \text{as} \quad k \to +\infty.$$

On the one hand, since $W_0^{1,\overrightarrow{p}(.)}(\Omega, \overrightarrow{w}(.)) \hookrightarrow\hookrightarrow L^{\underline{p}}(\Omega)$, then $u_k \to u$ in $L^{\underline{p}}(\Omega)$ for a subsequence denoted again $(u_k)_{k\in\mathbb{N}}$.

As $(u_k)_{k\in\mathbb{N}}$ is a bounded sequence in $W_0^{1,\overrightarrow{p}(.)}(\Omega, \overrightarrow{w}(.))$, by the growth condition $(a_i(x, T_n(u_k), \nabla u_k))_{k\in\mathbb{N}}$ is bounded in $L^{p_i'(.)}(\Omega, w_i^*(.))$, then, there exists a function $\phi_{i,n} \in L^{p_i'(.)}(\Omega, w_i^*(.))$ such that

$$a_i(x, T_n(u_k), \nabla u_k) \longrightarrow \varphi_{i,n} \quad \text{in} \quad L^{p_i'(.)}(\Omega, w_i^*(.)) \quad \text{as} \quad k \to \infty. \tag{4.4}$$

Of the same ways, we have $(g_n(x, u, \nabla u))_{k\in\mathbb{N}}$ is bounded in $L^{\underline{p}'}(\Omega)$ , then there exists a function $\varphi_n \in L^{\underline{p}'}(\Omega)$ such that

$$g_n(x, u, \nabla u) \longrightarrow \varphi_n \quad \text{in} \quad L^{\underline{p}'}(\Omega) \quad \text{as} \quad k \to \infty. \tag{4.5}$$

For all $v \in W_0^{1,\overrightarrow{p}(.)}(\Omega, \overrightarrow{w}(.))$ , we have

$$\begin{aligned} \langle \chi_n, v\rangle &= \lim_{k\to\infty} \sum_{i=1}^N \int_\Omega a_i(x, T_n(u_k), \nabla u_k) D^i v dx + \lim_{k\to\infty} \int_\Omega g_n(x, u_k, \nabla u_k) v dx \\ &= \sum_{i=1}^N \int_\Omega \phi_{i,n} D^i u dx + \int_\Omega \varphi_n v dx. \end{aligned} \tag{4.6}$$

From relations (4.3) and (4.6), we get

$$\begin{aligned}&\limsup_{k\to\infty}\langle S_n u_k,\, u_k\rangle\\&=\limsup_{k\to\infty}\sum_{i=1}^{N}\int_\Omega a_i(x,\, T_n(u_k),\, \nabla u_k)D^i u_k dx+\lim_{k\to\infty}\int_\Omega g_n(x,u_k,\nabla u_k)u_k dx\\&\leq\sum_{i=1}^{N}\int_\Omega \phi_{i,n}D^i u dx+\int_\Omega \varphi_n v dx.\end{aligned}\tag{4.7}$$

From to (4.5), we get

$$\int_\Omega g_n(x,\, u_k,\, \nabla u_k)u_k dx\longrightarrow\int_\Omega \varphi_n u dx.\tag{4.8}$$

Consequently

$$\limsup_{k\to\infty}\sum_{i=1}^{N}\int_\Omega a_i(x,\, T_n(u_k),\, \nabla u_k)D^i u_k dx\leq\sum_{i=1}^{N}\int_\Omega \phi_{i,n}D^i u dx.\tag{4.9}$$

On the other hand, by (1.5) we have

$$\sum_{i=1}^{N}\int_\Omega [a_i(x,\, T_n(u_k),\, \nabla u_k)-a_i(x,\, T_n(u_k),\, \nabla u)](D^i u_k-D^i u)dx\geq 0,\tag{4.10}$$

then,

$$\begin{aligned}&\sum_{i=1}^{N}\int_\Omega a_i(x,\, T_n(u_k),\, \nabla u_k)D^i u_k dx\\&\geq\sum_{i=1}^{N}\int_\Omega a_i(x,\, T_n(u_k),\, \nabla u_k)D^i u dx+\sum_{i=1}^{N}\int_\Omega a_i(x,\, T_n(u_k),\, \nabla u)(D^i u_k-D^i u)dx.\end{aligned}\tag{4.11}$$

In view of Lemma 3.3, we have $T_n(u_k)\to T(u)$ in $L^{p_i(.)}(\Omega,w_i)$, then

$$a_i(x,\, T_n(u_k),\, \nabla u)\longrightarrow a_i(x,\, T_n(u),\, \nabla u)\quad \text{in}\quad L^{p_i'(.)}(\Omega,w_i^*),$$

and from $(4.4)-(4.5)$ we obtain

$$\liminf_{k\to\infty}\sum_{i=1}^{N}\int_\Omega a_i(x,\, T_n(u_k),\, \nabla u_k)D^i u_k dx\geq\sum_{i=1}^{N}\int_\Omega \phi_{i,n}D^i u dx.\tag{4.12}$$

This implies by using (4.9) that

$$\lim_{k\to\infty}\sum_{i=1}^{N}\int_\Omega a_i(x,\, T_n(u_k),\, \nabla u_k)D^i u_k dx=\sum_{i=1}^{N}\int_\Omega \phi_{i,n}D^i u dx.\tag{4.13}$$

According to (4.6), (4.8) and (4.13), we get

$$\langle S_n u_k, u_k\rangle \longrightarrow \langle \chi_n, u\rangle \quad \text{as} \quad k\to\infty. \tag{4.14}$$

From (4.13) we will show that

$$\lim_{k\to+\infty}\sum_{i=1}^{N}\int_\Omega [a_i(x,\, T_n(u_k),\, \nabla u_k)-a_i(x,\, T_n(u_k),\, \nabla u)]\,(D^i u_k - D^i u)dx=0,$$

by virtue of Lemma 3.2, we get

$$u_k\longrightarrow u \quad \text{in} \quad W_0^{1,\overrightarrow{p}(.)}(\Omega,\, \overrightarrow{w}(.)) \quad \text{and} \quad D^i u_k \longrightarrow D^i u \quad \text{a.e. in} \quad \Omega.$$

Then,

$$a_i(x,\, T_n(u_k),\, \nabla u_k)\longrightarrow a_i(x,\, T_n(u),\, \nabla u) \quad \text{in } L^{p_i'(\cdot)}(\Omega, w_i^*(.)), \quad \text{for } i=1,\dots,N,$$
$$g_n(x,\, u_k,\, \nabla u_k)\to g_n(x,\, u,\, \nabla u) \quad \text{in} \quad L^{p_i'(\cdot)}(\Omega, w_i^*()).$$

Finally, we can conclude that $\chi_n = S_n u$.

It remains to show that $S_n$ is coercive, let $v_0\in K_\psi$, for all $v\in K_\psi$ we have

$$\begin{aligned}
\langle A_n v,\, v_0\rangle &= \sum_{i=1}^{N}\int_\Omega a_i(x,\, T_n(v),\, \nabla v)\, D^i v_0\, dx\\
&\le \sum_{i=1}^{N}\int_\Omega |a_i(x,\, T_n(v),\, \nabla v)|\, w_i^{\frac{1}{p_i(x)}}(x)\, w_i^{\frac{-1}{p_i(x)}}(x)\, D^i v_0\, dx\\
&\le 2\sum_{i=1}^{N}\left(\int_\Omega \left(w_i^{\frac{-1}{p_i(x)}}(x)\,|a_i(x,\, T_n(v),\, \nabla v)|\right)^{p_i'(x)} dx\right)^{\frac{1}{(p_i')^-}} \left\| w_i^{\frac{1}{p_i(x)}}(x)\, D^i v_0\right\|_{p_i(\cdot)}\\
&\le 2\sum_{i=1}^{N}\left(\int_\Omega w_i^*(x)\,(|a_i(x,\, T_n(v),\, \nabla v)|)^{p_i'(x)}\, dx\right)^{\frac{1}{(p_i')^-}} \|v_0\|_{1,\overrightarrow{p}(\cdot),\overrightarrow{w}(\cdot)}\\
&\le C_1\sum_{i=1}^{N}\left(\int_\Omega M_i^{p_i'(x)}(x) + n^{p_i(x)} + w_i(x)|D^i v|^{p_i(x)}dx + 1\right)^{\frac{1}{(p_i')^-}} \|v_0\|_{1,\overrightarrow{p}(\cdot),\overrightarrow{w}(\cdot)}
\end{aligned}$$

and

$$\langle A_n v,\, v\rangle = \sum_{i=1}^{N}\int_\Omega a_i(x,\, T_n(v),\, \nabla v)\, D^i v\, dx \ge \alpha\sum_{i=0}^{N}\int_\Omega w_i(x)|D^i v|^{p_i(x)}dx.$$

It follows that

$$\begin{aligned}
\frac{\langle A_n v, v-v_0\rangle}{\|v\|_{1,\overrightarrow{p}(.),\overrightarrow{w}(.)}} \ge{}& \frac{\alpha}{\|v\|_{1,\overrightarrow{p}(.),\overrightarrow{w}(.)}}\sum_{i=0}^{N}\int_\Omega w_i(x)|D^i v|^{p_i(x)}dx \\
&- \frac{C_1\|v_0\|_{1,\overrightarrow{p}(.),\overrightarrow{w}(.)}}{\|v\|_{1,\overrightarrow{p}(.),\overrightarrow{w}(.)}}\sum_{i=1}^{N}\left(\int_\Omega M_i^{p_i'(x)}(x) + n^{p_i(x)} + w_i(x)|D^i v|^{p_i(x)}dx + 1\right)^{\frac{1}{(p_i')^-}}\\
&\to\infty,
\end{aligned} \tag{4.15}$$

as $\|v\|_{1,\overrightarrow{p}(.),\overrightarrow{w}(.)} \to \infty$. On the other hand, thanks to (4.2), we have

$$\begin{aligned}\langle G_n v, v-v_0\rangle &= \int_\Omega g_n(x,\, v,\, \nabla v) v, dx - \int_\Omega g_n(x,\, v,\, \nabla v) v_0\, dx \\ &\geq -C'\left(\|v\|_{1,\overrightarrow{p}(.),\overrightarrow{w}(.)} + \|v_0\|_{1,\overrightarrow{p}(.),\overrightarrow{w}(.)}\right). \end{aligned} \tag{4.16}$$

We conclude that

$$\frac{\langle S_n v, v-v_0\rangle}{\|v\|_{1,\overrightarrow{p}(.),\overrightarrow{w}(.)}} = \frac{\langle A_n v, v-v_0\rangle}{\|v\|_{1,\overrightarrow{p}(.),\overrightarrow{w}(.)}} + \frac{\langle G_n v, v, -v_0\rangle}{\|v\|_{1,\overrightarrow{p}(.),\overrightarrow{w}(.)}} \longrightarrow \infty$$

$$\text{as } \|v\|_{1,\overrightarrow{p}(.),\overrightarrow{w}(.)} \to \infty.$$

□

The operator $S_n$ is pseudo-monotone and coercive. By virtue of ([14], Theorem 8.2), the approximate problem $(\mathcal{P}_n)$ has at least one solution $u_n \in W_0^{1,\overrightarrow{p}(.)}(\Omega,\, \overrightarrow{w}(.))$.

**Step 2 : A priori estimates**

Let $v = u_n - \eta \exp(G(|u_n|)) T_k(u)$ or $G(s) = \int_0^s b(t)/\alpha\; dt$ and $\eta \geq 0$. We then obtain that $v \in W_0^{1,\overrightarrow{p}(.)}(\Omega,\, \overrightarrow{w}(.))$, and for $\eta$ small enough, we deduce that $v \geq \psi$. Thus $v$ is an admissible test function in $(\mathcal{P}_n)$ and we have

$$\begin{aligned}&\sum_{i=1}^N \int_\Omega a_i(x,\, T_n(u_n),\, \nabla u_n) D^i(\exp(G(|u_n|)) T_k(u_n)) dx \\ &\quad + \int_\Omega g_n(x,\, u_n,\, \nabla u_n) \exp(G(|u_n)|) T_k(u_n) dx \\ &\leq \int_\Omega f_n \exp(G(|u_n|)) T_k(u_n) dx,\end{aligned}$$

which implies

$$\begin{aligned}&\sum_{i=1}^N \int_\Omega a_i(x,\, T_n(u_n),\, \nabla u_n) D^i u_n \frac{b(|u_n|)}{\alpha} \exp(G(|u_n|)) |T_k(u_n)| dx \\ &\quad + \sum_{i=1}^N \int_\Omega a_i(x,\, T_k(u_n),\, \nabla T_k(u_n)) D^i T_k(u_n) \exp(G(|u_n|)) dx \\ &\leq \int_\Omega |g_n(x,\, u_n,\, \nabla u_n)| \exp(G(|u_n|)) |T_k(u_n)| dx + \int_\Omega f_n \exp(G(|u_n|)) T_k(u_n) dx \\ &\leq \int_\Omega (|f_n| + |c(x)|) \exp(G(|u_n|)) |T_k(u_n)| dx \\ &\quad + \sum_{i=1}^N \int_\Omega b(|u_n|) |D^i u_n|^{p_i(x)} \exp(G(|u_n|)) |T_k(u_n)| dx.\end{aligned}$$

According to (1.3), we have

$$\begin{aligned}&\alpha\sum_{i=1}^{N}\int_{\Omega}|w_i(x)D^iT_k(u_n)|^{p_i(x)}dx\\ &\leq(\|f\|_{L^1(\Omega)}+\|c(x)\|_{L^1(\Omega)})\exp\left(\frac{\|b(\cdot)\|_{L^1(\mathbb{R})}}{\alpha}\right)k\leq C_1k,\end{aligned}$$

as a result

$$\sum_{i=0}^{N}\int_{\Omega}w_i(x)|D^iT_k(u_n)|^{p_i(x)}dx\leq C_2k\quad\text{ for }\quad k\geq 1,$$

where $C_2$ is a constant that does not depend on $n$. Then

$$\|T_k(u_n)\|_{1,\overrightarrow{p}(.),\overrightarrow{w}(.)}=\sum_{i=0}^{N}\|D^iT_k(u_n)\|_{p_i(.),w_i(.)}\leq C_3k^{\frac{1}{\underline{p}}}. \tag{4.17}$$

Now, we will show that

$$\lim_{j\to\infty}\sum_{i=1}^{N}\int_{\{j\leq|u_n|<j+1\}}a_i(x,\,u_n,\,\nabla u_n)D^iu_ndx=0. \tag{4.18}$$

Indeed, consider the function

$$v=u_n-\eta\exp(G(|u_n|))T_1(u_n-T_j(u_n))\ .$$

For $\eta$ small enough, we can obtain that $v\geq\psi$ and since $v\in W_0^{1,\overrightarrow{p}(.)}(\Omega,\,\overrightarrow{w}(.))$, then $v$ is an admissible test function in $(\mathcal{P}_n)$, which implies

$$\begin{aligned}&\sum_{i=1}^{N}\int_{\Omega}a_i(x,\,T_n(u_n),\,\nabla u_n)D^iT_1(u_n-T_j(u_n))\exp(G(|u_n|))dx\\ &\quad+\sum_{i=1}^{N}\int_{\Omega}a_i(x,\,T_n(u_n),\,\nabla u_n)D^iu_n\frac{b(|u_n|)}{\alpha}\exp(G(|u_n|))|T_1(u_n-T_j(u_n))|dx\\ &\quad+\int_{\Omega}g_n(x,\,u_n,\,\nabla u_n)\exp(G(|u_n|))T_1(u_n-T_j(u_n))dx\\ &\leq\int_{\Omega}f_n\exp(G(|u_n|))T_1(u_n-T_j(u_n))dx.\end{aligned}$$

In view of the growth condition (1.6), we deduce that

$$\begin{aligned}&\sum_{i=1}^{N}\int_{\{j\leq|u_n|<j+1\}}a_i(x,\,u_n,\,\nabla u_n)D^iu_n\exp(G(|u_n|))dx\\ &\leq\exp\left(\frac{\|b(\cdot)\|_{L^1(\mathbb{R})}}{\alpha}\right)\int_{\Omega}(|c(x)|+|f_n|)|T_1(u_n-T_j(u_n))|dx.\end{aligned}$$

Since $T_1(u_n-T_j(u_n))\longrightarrow 0$ weak$-\star$ in $L^\infty(\Omega)$ as $j$ goes to infinity, then the right-hand side tends to zero as $j$ goes to $\infty$. Then, we have (4.18).

**Step 3 : Weak convergence of truncations**

Firstly, we prove that $(u_n)_n$ is a Cauchy sequence in measure in $\Omega$.

Thanks to (4.17), Lemma 2.2 and the generalized Hölder's type inequality, we obtain

$$\begin{aligned} k\text{meas}\{|u_n|>k\} &= \int_{\{|u_n|>k\}} |T_k(u_n)|dx \leq \int_\Omega |T_k(u_n)|dx \\ &\leq \left(\frac{1}{p_0^-}+\frac{1}{(p_0')^-}\right)\|1\|_{p_0'(\cdot)}\|T_k(u_n)\|_{p_0(\cdot)} \\ &\leq 2\left(\text{meas}(\Omega)+1\right)^{\frac{1}{(p_0')^-}}\|T_k(u_n)\|_{1,\overrightarrow{p}(.),\overrightarrow{w}(.)} \\ &\leq C_4 k^{\frac{1}{\underline{p}}}, \end{aligned} \tag{4.19}$$

which implies that

$$\text{meas}\{|u_n|>k\} \leq C_4 \frac{1}{k^{1-\frac{1}{\underline{p}}}} \to 0 \quad \text{as} \quad k\to\infty. \tag{4.20}$$

For any $\lambda>0$, we have

$$\begin{aligned} &\text{meas}\{|u_n-u_m|>\lambda\} \\ &\leq \text{meas}\{|u_n|>k\}+\text{meas}\{|u_m|>k\}+\text{meas}\{|T_k(u_n)-T_k(u_m)|>\lambda\}. \end{aligned} \tag{4.21}$$

In view of (4.20), for all $\epsilon>0$, there exists $k_0>0$ such that

$$\text{meas}\{|u_n|>k\} \leq \frac{\epsilon}{3} \quad \text{and} \quad \text{meas}\{|u_m|>k\} \leq \frac{\epsilon}{3}, \quad \forall k \geq k_0. \tag{4.22}$$

On the other side, as the sequence $(T_k(u_n))_n$ is bounded in $W_0^{1,\overrightarrow{p}(.)}(\Omega, \overrightarrow{w}(.))$, there exists a subsequence still denoted $(T_k(u_n))_n$ such that

$$T_k(u_n) \rightharpoonup \mu_k \quad \text{in} \quad W_0^{1,\overrightarrow{p}(.)}(\Omega, \overrightarrow{w}(.)) \quad \text{as} \quad n\to\infty,$$

and by using the compact embedding, we deduce that

$$T_k(u_n) \to \mu_k \quad \text{in} \quad L^{\underline{p}}(\Omega) \qquad \text{and a.e in } \Omega,$$

then, we can suppose that $(T_k(u_n))_n$ is a Cauchy sequence in measure in $\Omega$. Thus, for all $k>0$ and $\lambda,\epsilon>0$, there exists $n_0=n_0(k, \lambda, \epsilon)$ such that

$$\text{meas}\{|T_k(u_n)-T_k(u_m)|>\lambda\} \leq \frac{\epsilon}{3} \quad \text{for all} \quad n,m \geq n_0(k, \lambda, \epsilon). \tag{4.23}$$

Using (4.22) and (4.23), we conclude that: for all $\lambda,\epsilon>0$, there exists $n_0=n_0(\lambda, \epsilon)$ such that

$$\text{meas}\{|u_n-u_m|>\lambda\} \leq \epsilon \quad \text{for any} \quad n,m \geq n_0(\lambda, \epsilon),$$

which implied that $(u_n)_n$ is a Cauchy sequence in measure, then converges almost everywhere, for a subsequence, to some measurable function $u$. Consequently, we obtain

$$T_k(u_n) \rightharpoonup T_k(u) \quad \text{in} \quad W_0^{1,\overrightarrow{p}(.)}(\Omega, \overrightarrow{w}(.)), \tag{4.24}$$

and using Lebesgue's dominated convergence theorem, we obtain

$$T_k(u_n) \to T_k(u) \quad \text{in} \quad L^{p_0(\cdot)}(\Omega). \tag{4.25}$$

**Step 4 : Convergence of the gradient**

Let $j \geq k > 0$ and $h_j(u_n) = 1 - |T_1(u_n - T_j(u_n))|$, we consider

$$v = u_n - \eta \exp(G(|u_n|))(T_k(u_n) - T_k(u))h_j(u_n).$$

Choose $\eta$ small enough such that $v \in K_\psi$. Since $h_j(u_n) = 1$ on $\{|u_n| \leq k\}$ and $T_k(u_n) - T_k(u)$ have the same sign as $u_n$ on the set $\{|u_n| > k\}$. Then, by taking $v$ as test function in $(\mathcal{P}_n)$, we have

$$\begin{aligned}
&\sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u_n), \nabla u_n) D^i(\exp(G(|u_n|))(T_k(u_n) - T_k(u))h_j(u_n))dx \\
&\qquad + \int_\Omega g_n(x, u_n, \nabla u_n) \exp(G(|u_n|))(T_k(u_n) - T_k(u))h_j(u_n)dx \\
&\leq \int_\Omega f_n \exp(G(|u_n|))(T_k(u_n) - T_k(u))h_j(u_n)dx.
\end{aligned} \tag{4.26}$$

According to (1.3) and (1.6), we obtain

$$\begin{aligned}
&\sum_{i=1}^{N} \int_{\{|u_n| \leq k\}} a_i(x, T_k(u_n), \nabla T_k(u_n))(D^i T_k(u_n) - D^i T_k(u)) \exp(G(|u_n|))dx \\
&\leq \int_\Omega (|c(x)| + |f_n|)|T_k(u_n) - T_k(u)| \exp(G(|u_n|))dx \\
&\quad + \sum_{i=1}^{N} \int_{\{k < |u_n| \leq j+1\}} |a_i(x, T_{j+1}(u_n), \nabla T_{j+1}(u_n))||D^j T_k(u)| \exp(G(|u_n|))dx \\
&\quad + \sum_{i=1}^{N} \int_{\{j < |u_n| \leq j+1\}} a_i(x, u_n, \nabla u_n) D^i u_n |T_k(u_n) - T_k(u)| \exp(G(|u_n|))dx.
\end{aligned} \tag{4.27}$$

Since $T_k(u_n) \to T_k(u)$ weak-$\star$ in $L^\infty(\Omega)$ , then the first term on the right-hand side of (4.27) approached to 0.

Since $\big(|a_i(x, T_{j+1}(u_n), \nabla T_{j+1}(u_n))|\big)_n$ is bounded in $L^{p_i'(\cdot)}(\Omega, w_i^*(.))$, there exists $\zeta_i \in L^{p_i'(\cdot)}(\Omega, w_i^*(.))$ such that

$$|a_i(x, T_{j+1}(u_n), \nabla T_{j+1}(u_n))| \longrightarrow \zeta_i \quad \text{in} \quad L^{p_i'(\cdot)}(\Omega, w_i^*(.)),$$

then, the second term on the right-hand side of (4.27), become

$$\begin{aligned}&\sum_{i=1}^{N}\int_{\{k<|u_n|\le j+1\}}|a_i(x,\,T_{j+1}(u_n),\,\nabla T_{j+1}(u_n))||D^iT_k(u)|\exp(G(|u_n|))dx\\ \le&\exp\left(\frac{\|b(\cdot)\|_{L^1(\mathbb{R})}}{\alpha}\right)\sum_{i=1}^{N}\int_{\{k<|u_n|\le j+1\}}|a_i(x,\,T_{j+1}(u_n),\,\nabla T_{j+1}(u_n))||D^iT_k(u)|dx\\ &\to\exp\left(\frac{\|b(\cdot)\|_{L^1(\mathbb{R})}}{\alpha}\right)\sum_{i=1}^{N}\int_{\{k<|u|\le j+1\}}\zeta_i|D^iT_k(u)|dx=0\quad\text{as}\quad n\to\infty,\end{aligned}\tag{4.28}$$

on the other hind, according (4.18), we have

$$\begin{aligned}&\sum_{i=1}^{N}\int_{\{j\le|u_n|\le j+1\}}a_i(x,\,u_n,\,\nabla u_n)D^iu_n|T_k(u_n)-T_k(u)|\exp(G(|u_n|))dx\\ \le&2k\exp\left(\frac{\|b(\cdot)\|_{L^1(\mathbb{R})}}{\alpha}\right)\sum_{i=1}^{N}\int_{\{j\le|u_n|\le j+1\}}a_i(x,\,u_n,\,\nabla u_n)D^iu_ndx\\ &\longrightarrow0\quad\text{as}\quad j\to\infty.\end{aligned}\tag{4.29}$$

Denote by $\epsilon_1(n),\epsilon_2(n),\ldots$ various functions of real numbers which converge to 0 as $n$ tends to infinity (respectively for $\epsilon_i(j,n)$).

By using (4.27)-(4.29), we obtain

$$\begin{aligned}&\sum_{i=1}^{N}\int_{\{|u_n|\le k\}}a_i(x,\,T_k(u_n),\,\nabla T_k(u_n))(D^iT_k(u_n)-D^iT_k(u))\exp(G(|u_n|))dx\\ \le&\epsilon_1(j,\,n),\end{aligned}\tag{4.30}$$

then,

$$\begin{aligned}&\sum_{i=1}^{N}\int_{\Omega}[a_i(x,\,T_k(u_n),\,\nabla T_k(u_n))-a_i(x,\,T_k(u_n),\,\nabla T_k(u))](D^iT_k(u_n)-D^iT_k(u))\exp(G(|u_n|))dx\\ \le&-\sum_{i=1}^{N}\int_{\Omega}a_i(x,\,T_k(u_n),\,\nabla T_k(u))(D^iT_k(u_n)-D^iT_k(u))\exp(G(|u_n|))dx\\ &-\sum_{i=1}^{N}\int_{\{|\mathrm{u}_n|>k\}}a_i(x,\,T_k(u_n),\,\nabla T_k(u_n))D^iT_k(u)\exp(G(|u_n|))dx+\epsilon_2(j,\,n).\end{aligned}\tag{4.31}$$

Using Lebesgue's dominated convergence theorem, we have $T_k(u_n)\longrightarrow T_k(u)$ in $L^{p_0}(\Omega)$ and since $D^iT_k(u_n)\longrightarrow D^iT_k(u)$ in $L^{p_i(\cdot)}(\Omega,w_i(.))$ , then, the terms on the right-hand side of (4.31) tends to 0 as $n$ goes to infinity. Therefore, we deduce that

$$\begin{aligned}&\sum_{i=1}^{N}\int_{\Omega}(a_i(x,\,T_k(u_n),\,\nabla T_k(u_n))-a_i(x,\,T_k(u_n),\,\nabla T_k(u)))(D^jT_k(u_n)-D^jT_k(u))dx\\ &\longrightarrow0\quad\text{as}\quad n\to\infty.\end{aligned}\tag{4.32}$$

In view to Lemma 3.2, we conclude that

$$T_k(u_n) \longrightarrow T_k(u) \quad \text{in } W_0^{1,\overrightarrow{p}(.)}(\Omega, \overrightarrow{p}(.)) \quad \text{and} \quad \nabla u_n \longrightarrow \nabla u \quad \text{a.e. in } \Omega. \tag{4.33}$$

**Step 5 : The equi-integrability of** $g_n(x, u_n, \nabla u_n)$

In this step, we will show that

$$g_n(x, u_n, \nabla u_n) \longrightarrow g(x, u, \nabla u) \quad \text{strongly in} \quad L^1(\Omega). \tag{4.34}$$

Let $v = u_n - \eta \exp(2G(|u_n|)) T_1(u_n - T_h(u_n))$, we obtain $v \in W_0^{1,\overrightarrow{p}(.)}(\Omega, \overrightarrow{w}(.))$, let $\eta$ small enough such that $v \geq \psi$, then $v$ is an admissible test function in $(\mathcal{P}_n)$, and we have

$$\begin{aligned} &\sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u_n), \nabla u_n) D^i(\exp(2G(|u_n|)) T_1(u_n - T_h(u_n))) dx \\ &\qquad + \int_\Omega g_n(x, u_n, \nabla u_n) \exp(2G(|u_n|)) T_1(u_n - T_h(u_n)) dx \\ &\leq \int_\Omega f_n \exp(2G(|u_n|)) T_1(u_n - T_h(u_n)) dx. \end{aligned}$$

Thanks to (1.3) and (1.6), we get

$$\begin{aligned} &\sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u_n), \nabla u_n) D^i u_n \frac{b(|u_n|)}{\alpha} \exp(2G(|u_n|)) |T_1(u_n - T_h(u_n))| dx \\ &\qquad + \sum_{i=1}^{N} \int_{\{h < |u_n| \leq h+1\}} a_i(x, u_n, \nabla u_n) D^i u_n \exp(2G(|u_n|)) dx \\ &\leq \exp\left( 2 \frac{\|b(\cdot)\|_{L^1(\mathbb{R})}}{\alpha} \right) \int_\Omega (|c(x)| + |f_n|) |T_1(u_n - T_h(u_n))| dx, \end{aligned}$$

witch implied that

$$\begin{aligned} &\sum_{i=1}^{N} \int_{\{h+1 \leq |u_n|\}} b(|u_n|) |D^i u_n|^{p_i(x)} w_i(x) dx \\ &\leq \exp\left( 2 \frac{\|b(\cdot)\|_{L^1(\mathbb{R})}}{\alpha} \right) \int_{\{h \leq |u_n|\}} (|c(x)| + |f|) dx, \end{aligned}$$

next, for all $\epsilon > 0$, there exists $h(\epsilon) > 0$ such that

$$\sum_{i=1}^{N} \int_{\{h+1 \leq |u_n|\}} b(|u_n|) |D^i u_n|^{p_i(x)} w_i(x) dx \leq \frac{\epsilon}{2}, \qquad \forall h \geq h(\epsilon). \tag{4.35}$$

On the other side, for each measurable subset $E\subset\Omega$, we have

$$\begin{aligned}&\sum_{i=1}^{N}\int_E b(|u_n|)|D^i u_n|^{p_i(x)}w_i(x)\,dx\\ &\leq \sum_{i=1}^{N}\int_E b(|T_{h+1}(u_n)|)|D^i T_{h+1}(u_n)|^{p_i(x)}w_i(x)\,dx\\ &\quad+\sum_{i=1}^{N}\int_{\{|u_n|\geq h+1\}} b(|u_n|)|D^i u_n|^{p_i(x)}w_i(x)\,dx.\end{aligned}\tag{4.36}$$

According to (4.33), there exists $\lambda(\epsilon)>0$ such that for any $E\subset\Omega$ with $\operatorname{meas}(E)\leq\lambda(\epsilon)$, we have

$$\sum_{i=1}^{N}\int_E b(|T_{h+1}(u_n)|)|D^i T_{h+1}(u_n)|^{p_i(x)}w_i(x)\,dx\leq\frac{\epsilon}{2}.\tag{4.37}$$

By using (4.35)–(4.37), we obtain for all $E\subset\Omega$ such that $\operatorname{meas}(E)\leq\lambda(\epsilon)$

$$\sum_{i=1}^{N}\int_E b(|u_n|)|D^i u_n|^{p_i(x)}w_i(x)\,dx\leq\epsilon,\tag{4.38}$$

thus in view of (1.6), the sequence $(g_n(x,\,u_n,\nabla u_n))_n$ is equi-integrability. Thanks to (4.33), we have

$$g_n(x,\,u_n,\nabla u_n)\to g(x,\,u,\,\nabla u)\qquad\text{a.e. in }\Omega.$$

Finally, thanks to Vitali's theorem, we prove (4.34).

**Step 6 : Passing to the limit**

Let $\varphi\in K_\psi\cap L^\infty(\Omega)$ and $M=k+\|\varphi\|_\infty$, and choosing $v=u_n-\eta T_k(u_n-\varphi)$ as a test function in $(\mathcal{P}_n)$, we get

$$\begin{cases}u_n\in K_\psi\\ \displaystyle\sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u_n),\nabla u_n)D^iT_k(u_n-\varphi)\,dx+\int_\Omega g_n(x,\,u_n,\,\nabla u_n)T_k(u_n-v)\,dx\\ \qquad\displaystyle\leq\int_\Omega f_n T_k(u_n-v)\,dx.\end{cases}$$

If $|u_n|>M$ then $|u_n-\varphi|\geq|u_n|-\|\varphi\|_\infty>k$, hence $\{|u_n-\varphi|\leq k\}\subseteq\{|u_n|\leq M\}$, which implies that

$$\begin{aligned}&\int_\Omega a_i(x,\,T_n(u_n),\,\nabla u_n)D^iT_k(u_n-\varphi)\,dx\\ &=\int_\Omega a_i(x,T_M(u_n),\nabla T_M(u_n))(D^iT_M(u_n)-D^i\varphi)\chi_{\{|u_n-\varphi|\leq k\}}\,dx\\ &=\int_\Omega(a_i(x,\,T_M(u_n),\,\nabla T_M(u_n))-a_i(x,\,T_M(u_n),\,\nabla\varphi))(D^iT_M(u_n)-D^i\varphi)\chi_{\{|u_n-\varphi|\leq k\}}\,dx\\ &\quad+\int_\Omega a_i(x,\,T_M(u_n),\,\nabla\varphi)(D^iT_M(u_n)-D^i\varphi)\chi_{\{|u_n-\varphi|\leq k\}}\,dx.\end{aligned}\tag{4.39}$$

By Fatou's Lemma, we obtain

$$\begin{aligned}&\liminf_{n\to+\infty}\int_\Omega a_i(x,T_n(u_n),\nabla u_n)D^iT_k(u_n-\varphi)dx\\ \geq&\int_\Omega (a_i(x,T_M(u),\nabla T_M(u))-a_i(x,T_M(u),\nabla\varphi))(D^iT_M(u)-D^i\varphi)\chi_{\{|u-\varphi|\leq k\}}dx\\ &+\lim_{n\to\infty}\int_\Omega a_i(x,T_M(u_n),\nabla\varphi)(D^iT_M(u_n)-D^i\varphi)\chi_{\{|u_n-\varphi|\leq k\}}dx. \end{aligned}\tag{4.40}$$

The second term on the right-hand side of (4.40) is equal to

$$\int_\Omega a_i(x,T_M(u),\nabla\varphi)(D^iT_M(u)-D^i\varphi)\chi_{\{|u-\varphi|\leq k\}}dx.$$

Consequently

$$\begin{aligned}&\liminf_{n\to\infty}\sum_{i=1}^N\int_\Omega a_i(x,T_n(u_n),\nabla u_n)D^iT_k(u_n-\varphi)dx\\ \geq&\sum_{i=1}^N\int_\Omega a_i(x,T_M(u),\nabla T_M(u))(D^iT_M(u)-D^i\varphi)\chi_{\{|u-\varphi|\leq k\}}dx\\ =&\sum_{i=1}^N\int_\Omega a_i(x,u,\nabla u)D^iT_k(u-\varphi)dx. \end{aligned}\tag{4.41}$$

On the other side, we have $T_k(u_n-\varphi)\longrightarrow T_k(u-\varphi)$ weak $-\star$ in $L^\infty(\Omega)$ and using (4.34), we obtain

$$\int_\Omega g_n(x,u_n,\nabla u_n)T_k(u_n-\varphi)dx\to\int_\Omega g(x,u,\nabla u)T_k(u-\varphi)dx,\tag{4.42}$$

and

$$\int_\Omega f_nT_k(u_n-\varphi)dx\to\int_\Omega fT_k(u-\varphi)dx.\tag{4.43}$$

Finally, the proof of Theorem 4.1 is complete. □

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