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On the h-almost Yamabe Soliton

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Abstract. We introduce the concept h-almost Yamabe soliton which extends naturally the almost Yamabe soliton by Barbosa-Ribeiro and obtain some rigidity results concerning h-almost Yamabe solitons. Some condition for a compact h-almost Yamabe soliton to be a gradient soliton is also obtained. Finally, we give some characterizations for a special class of gradient h-almost Yamabe solitons.

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1 Introduction

The Yamabe flow was introduced by Hamilton at the same time as the Ricci flow as an attempt to solve the Yamabe problem on manifolds of positive conformal Yamabe invariant. Formally, the Yamabe flow deforms a given manifold by evolving its metric according to

$$\frac{\partial}{\partial t}g(t)=-R(t)g(t), \tag{1.1}$$

where $R(t)$ denotes the scalar curvature of the metric $g(t)$. The Yamabe flow and the Ricci flow are equivalent in dimension $n=2$, but they are essentially different in higher dimensions [1].

A family of metrics $g(t)=\sigma(t)\psi_t^*g(0)$ solving (1.1), where $\sigma(t)$ is a positive smooth function and $\psi_t:M\to M$ is a one-parameter family of diffeomorphisms of M, is said to be a self-similar solution of the Yamabe flow. Yamabe solitons are self-similar solutions of the Yamabe flow. A Riemannian manifold (M^n,g) is a Yamabe soliton if it admits a vector field X such that

$$(R-\rho)g=\frac{1}{2}\mathcal{L}_Xg, \tag{1.2}$$

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where $\mathcal{L}_X$ denotes the Lie derivative in the direction of the vector field X and ρ is a real number. For $\rho=0$ the Yamabe soliton is steady, for $\rho<0$ is expanding, and for $\rho>0$ is shrinking. It has been known (see [2]) that a compact Yamabe soliton has constant scalar curvature, thus trivial. For more details on Yamabe soliton we refer the reader to [3–5].

Barbosa and Ribeiro introduced the almost Yamabe soliton in [6] as follows. A Riemannian manifold (M^n,g) is an almost Yamabe soliton if there exists a complete vector field X and a smooth soliton function ρ on (M^n,g) satisfying

$$(R-\rho)g=\frac{1}{2}\mathcal{L}_X g. \tag{1.3}$$

From the definition, if ρ is constant, almost Yamabe solitons are Yamabe solitons.

Recently, Gomes, Wang and Xia [7] have introduced the definition of the h-almost Ricci soliton. Such a soliton is a generalization of an almost Ricci soliton given by the authors in [8]. An h-almost Ricci soliton is a complete Riemannian manifold (M^n,g) with a vector field $X\in\mathfrak{X}(M)$, a soliton function $\lambda:M\to\mathbb{R}$ and a function $h:M\to\mathbb{R}$ which are smooth and satisfy the equation

$$Ric+\frac{h}{2}\mathcal{L}_X g=\lambda g. \tag{1.4}$$

In particular, the results contained in [7, 9] indicates that h-almost Ricci solitons should reveal a reasonably broad generalization of the fruitful concept of the classical Ricci soliton, almost Ricci soliton and quasi Einstein manifolds [10–14]. For further details about h-almost Ricci soliton, see [7, 9].

Therefore, it is very interesting to consider a generalization of the almost Yamabe solitons.

Definition 1.1. We say that a Riemannian manifold (M^n,g) is an h-almost Yamabe soliton if there exists a complete vector field X, a smooth soliton function $\rho:M\to\mathbb{R}$ and a signal function $h:M\to\mathbb{R}$ which satisfy the equation:

$$(R-\rho)g=\frac{h}{2}\mathcal{L}_X g, \tag{1.5}$$

where R denotes the scalar curvature of M^n. The function h is said to have a defined signal if either $h>0$ or $h<0$ on M.

In the following, we denote the h-almost Yamabe soliton satisfying (1.5) by (M^n,g,X,h,ρ). When ρ is constant the structure is said to be an h-Yamabe soliton. Moreover, if $X=\nabla u$, we call the equation

$$(R-\rho)g=\frac{h}{2}\mathcal{L}_{\nabla u} g, \tag{1.6}$$

a gradient h-almost Yamabe soliton, where $\nabla^2 u$ denotes the Hessian of u. An h-almost Yamabe soliton is said to be shrinking, steady or expanding if it admits a soliton field for which, respectively, $\rho>0$, $\rho=0$ or $\rho<0$. Otherwise, it will be called indefinite.

Moreover, when either the vector field X is trivial or the potential u is constant, an h-almost Yamabe soliton will be called *trivial*, otherwise it will be a *nontrivial* h-almost Yamabe soliton. Let us point out that the traditional Yamabe soliton is a 1-Yamabe soliton with constant ρ. Moreover, 1-almost Yamabe soliton is just the almost Yamabe soliton. Some interesting examples of 1-almost Yamabe soliton are given in [6].

Inspired by [15], Leandro and Pina [16] have introduced the definition of generalized quasi Yamabe gradient soliton, more precisely, we say that a complete Riemannian manifold (M^n,g) with $n\geq 3$ is a generalized quasi Yamabe gradient soliton, if there exists a constant ρ and two functions f and μ defined on M such that

$$(R-\rho)g=\nabla^2 f-\mu df\otimes df, \tag{1.7}$$

where df is the dual 1-form of ∇f. When $\mu=\frac{1}{m}$, where m is not zero, the above generalized quasi Yamabe gradient soliton is called a quasi Yamabe gradient soliton (see [17]) which will be denoted by (M^n,g,f). It has been proved in [17] that locally conformally flat quasi Yamabe gradient solitons with positive sectional curvature are rotationally symmetric. Moreover, they proved that a compact quasi Yamabe gradient soliton has constant scalar curvature. Wang [18] gave several estimates for the scalar curvature and the potential function of the quasi Yamabe gradient solitons. In [19, 20], they defined the almost quasi Yamabe solitons

$$(R-\rho)g=\nabla^2 f-\frac{1}{m}df\otimes df, \tag{1.8}$$

where ρ is a function. They also obtained some conditions for an almost quasi Yamabe soliton to be trivial and some necessary and sufficient conditions for a compact almost quasi Yamabe soliton to be a gradient soliton.

Now we consider a nonconstant function $u=e^{-\frac{f}{m}}$. We have

$$\nabla u=-\frac{1}{m}e^{-\frac{f}{m}}\nabla f \quad \text{and} \quad -\frac{m}{u}\nabla^2 u=\nabla^2 f-\frac{1}{m}df\otimes df. \tag{1.9}$$

Consequently, Equation (1.8) can be rewritten as

$$(R-\rho)g=-\frac{m}{2u}\mathcal{L}_{\nabla u}g, \tag{1.10}$$

which implies that all almost quasi Yamabe solitons are a gradient $(-\frac{m}{u})$-almost Yamabe soliton.

From the equivalence between Eqs. (1.8) and (1.10), we have the following example.

Example 1.1. Let $(\mathbb{R}^n,g_{euc})$ be the Euclidean space. For a nonzero real number m and a positive constant β, the functions

$$f=-m\ln u,\ u=||x||^2+\beta \text{ and } \rho=\frac{2m}{u}$$

parameterize $(\mathbb{R}^n, g_{euc})$ with a nontrivial gradient $(-\frac{m}{u})$-almost Yamabe soliton. Indeed, since $\mathbb{R}^n$ has null scalar curvature and $\nabla^2||x||^2 = 2g_{euc}$, we conclude our statement by straightforward computation from Eq. (1.10).

In this paper, We treat the Yamabe soliton equation, the almost Yamabe soliton equation and almost quasi Yamabe soliton equation in a unified way, and it is interested to investigate whether the h-almost Yamabe solitons share a similar property. For what follows we assume that (M^n, g) has dimension $n \geq 3$ and that h has defined signal.

In the following we show that, with natural conditions and non-positive Ricci curvature, any complete h-almost Yamabe soliton is trivial.

Theorem 1.1. *Suppose (M^n, g, X, h, ρ) is an h-almost Yamabe soliton with non-positive Ricci curvature such that*

$$\int_M d(x, x_0)^{-2}|X|^2 dv < \infty. \tag{1.11}$$

Then (M^n, g, X, h, ρ) is a trivial h-almost Yamabe soliton. Here $d(x, x_0)$ denotes the distance from x to x_0 with respect to the Riemannian metric g.

We remark that we do not assume in Theorem 1.1 that the h-almost Yamabe soliton (M^n, g) is either gradient or compact. Note also that the corresponding theorem for the Yamabe soliton was proved by [21].

The next result gives some integral conditions which force h-almost Yamabe soliton to be trival. Its proof is motivated by the corresponding result for almost Yamabe solitons proven in Barbosa-Ribeiro [6].

Theorem 1.2. *Let (M^n, g, X, h, ρ) be an h-almost Yamabe soliton. Then (M^n, g, X, h, ρ) is trivial if one of the following conditions holds:*

1. *(M^n, g) is compact and ρ is a constant;*
2. *(M^n, g) is compact and $\int_M \langle \nabla(h(R-\rho)), X\rangle dv \geq 0$;*
3. *(M^n, g) is compact and $\int_M \{Ric(X, X) + (n-2)\langle \nabla(h^{-1}(R-\rho)), X\rangle\} dv \leq 0$;*
4. *$|X| \in \mathcal{L}^1(M)$ and either $R \geq \rho$ or $R \leq \rho$.*

We remark that (1.5) implies that the associated vector field X is a conformal vector field with conformal factor $h^{-1}(R-\rho)$. Therefore, Theorems 1.1 and 1.2 give some conditions that X is a Killing vector field, i.e., $R = \rho$.

On the other hand, a conformal vector field X is called closed when it satisfies the following condition:

$$\nabla_Y X = \rho Y,$$

for all $Y\in\mathfrak{X}(M)$. Obviously, a gradient conformal vector field $X=\nabla u$ is closed. When we have a compact h-almost Yamabe soliton, we can use the Hodge-de Rham decomposition theorem to write

$$X=\nabla\varphi+Y, \tag{1.12}$$

where $Y\in\mathfrak{X}(M)$ satisfies $divY=0$, whereas φ is a smooth function on (M^n,g). The following theorem gives some conditions for a compact h-almost Yamabe soliton to be a gradient soliton.

Theorem 1.3. *Let (M^n,g,X,h,ρ) be a compact h-almost Yamabe soliton. The following statements hold:*

1. *If it is also a gradient h-almost Yamabe soliton with potential u, then, up to a constant, u agrees with the Hodge-de Rham potential φ.*

2. *The compact nontrivial h-almost Yamabe soliton of constant scalar curvature with closed conformal vector field X vanishing at some point of (M^n,g) admits a gradient structure.*

3. *The compact h-almost Yamabe soliton (M^n,g,X,h,ρ) is gradiant if and only if*

$$\int_M Ric(\nabla\varphi,Y)dv\leq 0.$$

We point out that Theorem 1.3 is an extension of the almost Yamabe soliton case and almost quasi-Yamabe soliton case proved recently in [6,20] to the h-almost Yamabe soliton case.

The last two theorems give strong restrictions to a special class of gradient h-almost Yamabe soliton. Its proof is motivated by [22] and [20].

Theorem 1.4. *Let u be a positive function on (M^n,g) and m a nonzero constant. If $(M^n,g,\nabla u,-\frac{m}{u},\rho)$ is a compact gradient $(-\frac{m}{u})$-almost Yamabe soliton with constant scalar curvature such that its first nonzero eigenvalue λ_1 satisfies*

$$\lambda_1>\frac{R}{n-1}, \tag{1.13}$$

then $(M^n,g,\nabla u,-\frac{m}{u},\rho)$ is a trivial $(-\frac{m}{u})$-almost Yamabe soliton.

Theorem 1.5. *Let u be a positive function on (M^n,g) and m a nonzero constant. For a gradient $(-\frac{m}{u})$-almost Yamabe soliton $(M^n,g,\nabla u,-\frac{m}{u},\rho)$, the following is valid:*

1. *If (M^n,g) is complete and $(R-\rho)$ is a nonzero constant, then (M^n,g) is isometric with $\mathbb{S}^n\left(\sqrt{\frac{R-\rho}{m}}\right)$;*

2. *If (M^n,g) is compact and R is a constant, then (M^n,g) is isometric to a Euclidean sphere.*

2 Preliminaries

Lemma 2.1. *Let (M^n,g,X,h,ρ) be an h-almost Yamabe soliton. Then one can get*

$$divX=nh^{-1}(R-\rho), \tag{2.1}$$

$$(2-n)\nabla_X(h^{-1}(R-\rho))=\frac{1}{2}\Delta|X|^2-|\nabla X|^2+Ric(X,X). \tag{2.2}$$

Proof. Taking the trace of (1.5), we get (2.1). Moreover, we have the following Bochner formula (see [23]):

$$div(\mathcal{L}_X g)(X)=\frac{1}{2}\Delta|X|^2-|\nabla X|^2+Ric(X,X)+\nabla_X divX. \tag{2.3}$$

From (1.5), we have

$$\mathcal{L}_X g=2h^{-1}(R-\rho)g. \tag{2.4}$$

Substituting (2.1) and (2.4) into (2.3), we obtain

$$\nabla_X(2h^{-1}(R-\rho))=\frac{1}{2}\Delta|X|^2-|\nabla X|^2+Ric(X,X)+\nabla_X(nh^{-1}(R-\rho)),$$

or equivalently,

$$(2-n)\nabla_X(h^{-1}(R-\rho))=\frac{1}{2}\Delta|X|^2-|\nabla X|^2+Ric(X,X),$$

which is the desired (2.2). □

Lemma 2.2. *Let $(M^n,g,\nabla u,-\frac{m}{u},\rho)$ be a gradient $(-\frac{m}{u})$-almost Yamabe soliton. Then one can get*

$$\Delta u=-\frac{n}{m}(R-\rho)u, \tag{2.5}$$

$$\frac{n-1}{m}\nabla((R-\rho)u)=Ric(\nabla u), \tag{2.6}$$

$$\frac{n-1}{m}\Delta((R-\rho)u)=\frac{1}{2}\langle\nabla u,\nabla R\rangle-\frac{1}{m}R(R-\rho)u, \tag{2.7}$$

$$\frac{1}{2}\Delta|\nabla u|^2-|\nabla^2 u|^2+\frac{1}{n-1}Ric(\nabla u,\nabla u)=0. \tag{2.8}$$

Proof. We calculate by using the local orthonormal frame. Taking the trace of (1.10) yields

$$-\frac{m}{u}\Delta u=n(R-\rho),$$

or equivalently,

$$\Delta u=-\frac{n}{m}(R-\rho)u.$$

Hence, we obtain (2.5).

Furthermore, it follows from (1.10) that

$$u_{ij} = -\frac{u}{m}(R-\rho)g_{ij}. \tag{2.9}$$

Differentiating (2.9) leads to

$$u_{ijk} = \nabla_k \left(-\frac{u}{m}(R-\rho) \right) g_{ij}.$$

Hence

$$u_{iji} = \nabla_j \left(-\frac{u}{m}(R-\rho) \right).$$

By using the Ricci identity

$$u_{iji} = \nabla_j \Delta u + R_{ij} u_i$$

and (2.5), we have

$$\frac{n-1}{m} \nabla_j ((R-\rho)u) = R_{ij} u_i.$$

Hence, we obtain (2.6).

Differentiating (2.6) again, we have

$$\frac{n-1}{m} \nabla_k \nabla_j ((R-\rho)u) = \nabla_k R_{ij} u_i + R_{ij} u_{ik}. \tag{2.10}$$

Tracing (2.10) and using the fact that $\nabla_j R_{ij} = \frac{1}{2} \nabla_i R$ yields

$$\frac{n-1}{m} \Delta ((R-\rho)u) = \frac{1}{2} \nabla_i R u_i - \frac{1}{m} R(R-\rho)u.$$

Hence, we obtain (2.7).

Substituting $X = \nabla u$ and $h = -\frac{m}{u}$ into (2.2) yields

$$\frac{1}{2}\Delta |\nabla u|^2 - |\nabla^2 u|^2 + Ric(\nabla u, \nabla u) = \frac{n-2}{m} \langle \nabla u, \nabla (R-\rho)u \rangle. \tag{2.11}$$

By (2.6), we have

$$\frac{1}{2}\Delta |\nabla u|^2 - |\nabla^2 u|^2 + \frac{1}{n-1} Ric(\nabla u, \nabla u) = 0.$$

Hence, we obtain (2.8). □

3 Proof of Theorem 1.1

Consider a cut-off function ϕ such that $\phi \in C_0^2(\mathbb{B}_{2r}(x_0))$, $0 \le \phi \le 1$,

$$\phi = 1, \quad in \quad \mathbb{B}_r(x_0); \qquad |\nabla \phi|^2 \le \frac{C}{r^2}$$

and

$$\Delta \phi \le \frac{C}{r^2}.$$

These imply that

$$\Delta \phi^2 \le \frac{C}{r^2} \to 0, \quad as \quad r \to \infty. \tag{3.1}$$

From (2.2), we have

$$|\nabla X|^2 = \frac{1}{2}\Delta |X|^2 + Ric(X,X) + (n-2)\nabla_X(h^{-1}(R-\rho)). \tag{3.2}$$

Multiplying ϕ^2 to (3.2) and integrating it over (M^n, g), we get

$$\begin{aligned} &\int_M |\nabla X|^2 \phi^2 dv \\ =&\frac{1}{2}\int_M |X|^2 \Delta \phi^2 dv + \int_M Ric(X,X)\phi^2 dv + (n-2)\int_M \phi^2 \nabla_X(h^{-1}(R-\rho))dv. \end{aligned} \tag{3.3}$$

On the other hand, by integration by parts and (2.1), we have

$$\begin{aligned} &\int_M \phi^2 \nabla_X (h^{-1}(R-\rho))dv \\ =&-\int_M (h^{-1}(R-\rho))\phi^2 divX dv - 2\int_M (h^{-1}(R-\rho))\phi \nabla_X \phi dv \\ =&-n\int_M (h^{-1}(R-\rho))^2 \phi^2 dv - 2\int_M (h^{-1}(R-\rho))\phi \nabla_X \phi dv. \end{aligned} \tag{3.4}$$

Substituting (3.4) into (3.3), we get

$$\begin{aligned} &\int_M |\nabla X|^2 \phi^2 dv + n(n-2)\int_M (h^{-1}(R-\rho))^2 \phi^2 dv \\ =&\frac{1}{2}\int_M |X|^2 \Delta \phi^2 dv + \int_M Ric(X,X)\phi^2 dv - 2(n-2)\int_M (h^{-1}(R-\rho))\phi \nabla_X \phi dv. \end{aligned} \tag{3.5}$$

Applying the Cauchy-Schwarz's inequality and the Young's inequality, we obtain

$$\begin{aligned}
&-2(n-2)\int_M (h^{-1}(R-\rho))\phi\nabla_X\phi dv\\
=&2(n-2)\int_M (h^{-1}(\rho-R))\phi\nabla_X\phi dv\\
\leq&2(n-2)\left(\int_M \left|(h^{-1}(\rho-R))\phi\right|^2 dv\right)^{\frac{1}{2}}\left(\int_M |\nabla_X\phi|^2 dv\right)^{\frac{1}{2}}\\
\leq&(n-2)\left(\int_M \left|(h^{-1}(\rho-R))\phi\right|^2 dv+\int_M |\nabla_X\phi|^2 dv\right)\\
\leq&(n-2)\left(\int_M \left|(h^{-1}(\rho-R))\phi\right|^2 dv+\int_M |X|^2|\nabla\phi|^2 dv\right).
\end{aligned}\tag{3.6}$$

Substituting (3.6) into (3.5), we get

$$\begin{aligned}
&\int_M |\nabla X|^2\phi^2 dv+(n-1)(n-2)\int_M (h^{-1}(R-\rho))^2\phi^2 dv-\int_M Ric(X,X)\phi^2 dv\\
\leq&\frac{1}{2}\int_M |X|^2\Delta\phi^2 dv+(n-2)\int_M |X|^2|\nabla\phi|^2 dv.
\end{aligned}\tag{3.7}$$

In view of (1.11), (3.1) and the assumption that (M^n,g) has non-positive Ricci curvature, we can conclude that the right hand side of (3.7) tends to zero as $r\to\infty$. So we can conclude that the left hand side of (3.7) tends to zero as $r\to\infty$, which implies $(h^{-1}(R-\rho))^2=0$ in (M^n,g). Hence Theorem 1.1 follows. □

4 Proof of Theorem 1.2

The following lemmas play crucial roles in this section:

Lemma 4.1 ([24]). *Let T be a symmetric $(0,2)$-tensor on a Riemannian manifold (M^n,g) and ψ a smooth function on (M^n,g). Then we have*

$$div(T(\psi Z))=\psi(divT)(Z)+\psi\langle T,\nabla Z\rangle+T(\nabla\psi,Z),$$

for all $Z\in\mathfrak{X}(M)$.

Lemma 4.2 ([25]). *For any conformal vector field X on a compact Riemannian manifold (M^n,g), the following identity holds*

$$\int_M \langle\nabla R,X\rangle dv=0.$$

Lemma 4.3. ([26]) *Let X be a smooth vector field on the complete, noncompact, oriented Riemannian manifold (M^n,g), such that $divX$ doesn't change sign on (M^n,g). If $|X|\in L^1(M)$, then $divX=0$ on (M^n,g).*

Now we prove Theorem 1.2.

Proof of Theorem 1.2. (1) Setting $\tilde{Ric}:=Ric-\frac{\rho}{n}g$ and taking $T=\tilde{Ric}$, $\psi=1$ and $Z=X$ in Lemma 4.1, we obtain

$$\begin{aligned} div(\tilde{Ric}(X)) &= (div\tilde{Ric})(X)+\langle \tilde{Ric},\nabla X\rangle \\ &= (div\tilde{Ric})(X)+\left\langle \tilde{Ric},\frac{1}{2}\mathcal{L}_X g\right\rangle \\ &= \left\langle \frac{1}{2}\nabla R-\frac{1}{n}\nabla\rho,X\right\rangle+h^{-1}(R-\rho)^2, \end{aligned} \tag{4.1}$$

where we have used (2.4) and $\nabla_j R_{ij}=\frac{1}{2}\nabla_i R$ in the last equality. By integrating the above equation, we obtain

$$\int_M \left(\frac{1}{2}\langle\nabla R,X\rangle-\frac{1}{n}\langle\nabla\rho,X\rangle+h^{-1}(R-\rho)^2\right)dv=0.$$

Since X is a conformal vector field, by Lemma 4.2, we have

$$\int_M \langle\nabla R,X\rangle dv=0,$$

which gives

$$\int_M \left\{-\frac{1}{n}\langle\nabla\rho,X\rangle+h^{-1}(R-\rho)^2\right\}dv=0. \tag{4.2}$$

If ρ is a constant, then (M^n,g) must be a trivial h-almost Yamabe soliton and has constant scalar curvature.

(2) By integration by parts and (2.1), we have

$$\int_M (R-\rho)^2 dv=\int_M \frac{1}{n}h(R-\rho)divX dv=-\frac{1}{n}\int_M \langle\nabla(h(R-\rho)),X\rangle dv.$$

Thus, we may use the assumption to deduce that $R=\rho$, and from the fundamental equation, (M^n,g) must be a trivial h-almost Yamabe soliton.

(3) By integrating (2.2), we have

$$\int_M |\nabla X|^2 dv=\int_M \{Ric(X,X)+(n-2)\nabla_X(h^{-1}(R-\rho))\}dv.$$

Since we are assuming that the right hand side is less than or equal to zero, we obtain $\nabla X = 0$, therefore, $\mathcal{L}_X g = 0$. Hence, (M^n, g) is a trivial h-almost Yamabe soliton.

(4) We use Lemma 4.3 to deduce from (2.1) that $divX = 0$, therefore, $R = \rho$ which implies that (M^n, g) must be a trivial h-almost Yamabe soliton. We complete the proof of the theorem. □

5 Proof of Theorem 1.3

The following lemmas play crucial roles in this section:

Lemma 5.1 ([27]). *Let X be a nontrivial conformal vector field on a compact Riemannian manifold (M^n, g) of constant scalar curvature R. Then $R > 0$.*

Lemma 5.2 ([28]). *Let (M^n, g) be a connected compact Riemannian manifold with the constant curvature $R > 0$. Then (M^n, g) is globally isometric to a sphere if (M^n, g) admits a closed conformal vector field X which vanishes at some point of (M^n, g).*

Now we prove Theorem 1.3.

Proof of Theorem 1.3. (1) First we notice that for a compact h-almost Yamabe soliton (M^n, g, X, h, ρ) it follows that

$$divX = nh^{-1}(R - \rho).$$

Now using the Hodge-de Rham decomposition we have $divX = \Delta\varphi$, we get

$$\Delta\varphi = nh^{-1}(R - \rho). \tag{5.1}$$

On the other hand, if $(M^n, g, \nabla u, h, \rho)$ is also a compact gradient h-almost Yamabe soliton, then we have

$$\Delta u = nh^{-1}(R - \rho). \tag{5.2}$$

Comparing Eqs. (5.1) and (5.2) we deduce $\Delta(u - \varphi) = 0$. Now it is enough to use Hopf's theorem to conclude $u = \varphi + c$.

(2) Since (M^n, g) is compact with constant scalar curvature R and X is a nontrivial closed conformal vector field vanishing at some point of (M^n, g), using Lemmas 5.1 and 5.2, we conclude that (M^n, g) is isometric to a Euclidean sphere $\mathbb{S}^n$. Thus, (M^n, g) is an Einstein Riemannian manifold, i.e.,

$$Ric = \frac{1}{n} Rg.$$

Recall that for a compact h-almost Yamabe soliton (M^n, g, X, h, ρ) it follows that

$$(R - \rho)g = \frac{h}{2}\mathcal{L}_X g.$$

Comparing the above two equations we have that $\left(M^n,g,X,h,\lambda=\frac{(n+1)}{n}R-\rho\right)$ is a compact nontrivial h-almost Ricci soliton with constant scalar curvature:

$$Ric+\frac{h}{2}\mathcal{L}_X g=\lambda g.$$

Hence, from [7] we conclude that (M^n,g) admits a gradient structure.

(3) Using (1.12) and (2.4), we have

$$\frac{1}{2}\mathcal{L}_Y g=h^{-1}(R-\rho)g-\nabla^2\varphi, \tag{5.3}$$

which gives

$$\Delta\varphi=nh^{-1}(R-\rho)$$

and

$$\begin{aligned}\int\limits_M\left|\frac{1}{2}\mathcal{L}_Y g\right|^2 dv&=\int\limits_M\left\{n(h^{-1}(R-\rho))^2+|\nabla^2\varphi|^2-2\langle\nabla^2\varphi,h^{-1}(R-\rho)g\rangle\right\}dv\\&=\int\limits_M\left\{-n(h^{-1}(R-\rho))^2+|\nabla^2\varphi|^2\right\}dv.\end{aligned} \tag{5.4}$$

Taking divergence of the equation (5.3) and using (2.3), we have

$$\begin{aligned}&\frac{1}{2}\Delta|Y|^2-|\nabla Y|^2+Ric(Y,Y)\\&=2\langle Y,\nabla(h^{-1}(R-\rho))\rangle-2\langle Y,\Delta\nabla\varphi\rangle\\&=2\langle Y,\nabla(h^{-1}(R-\rho))\rangle-2\langle Y,\nabla\Delta\varphi\rangle-2Ric(\nabla\varphi,Y).\end{aligned} \tag{5.5}$$

Integrating (5.5) and using the integration by parts formula, we have

$$\begin{aligned}\int\limits_M|\nabla Y|^2 dv&=\int\limits_M\{Ric(Y,Y)+2Ric(\nabla\varphi,Y)\}dv\\&=\int\limits_M\{Ric(X,X)-Ric(\nabla\varphi,\nabla\varphi)\}dv.\end{aligned} \tag{5.6}$$

Integrating (2.2) and using the integration by parts formula, we have

$$\begin{aligned}\int\limits_M Ric(X,X)dv&=\int\limits_M\{|\nabla X|^2-(n-2)\nabla_X(h^{-1}(R-\rho))\}dv\\&=\int\limits_M\{|\nabla X|^2+n(n-2)(h^{-1}(R-\rho))^2\}dv.\end{aligned} \tag{5.7}$$

Next, integrating Bochner's formula and using the integration by parts formula yields

$$\begin{aligned}\int_M Ric(\nabla\varphi,\nabla\varphi)dv &= \int_M \{-|\nabla^2\varphi|^2 - \langle\nabla\varphi,\nabla\Delta\varphi\rangle\}dv \\ &= \int_M \{-|\nabla^2\varphi|^2 + (\Delta\varphi)^2\}dv \\ &= \int_M \{-|\nabla^2\varphi|^2 + n^2(h^{-1}(R-\rho))^2\}dv. \end{aligned} \tag{5.8}$$

On the other hand, a simple calculation yields

$$\begin{aligned}\int_M \{|\nabla X|^2 - |\nabla Y|^2\}dv &= \int_M \{|\nabla^2\varphi|^2 + 2\langle\nabla^2\varphi,\nabla Y\rangle\}dv \\ &= \int_M \{|\nabla^2\varphi|^2 - 2\langle\Delta\nabla\varphi, Y\rangle\}dv \\ &= \int_M \{|\nabla^2\varphi|^2 - 2\langle\nabla\Delta\varphi + Ric(\nabla\varphi), Y\rangle\}dv \\ &= \int_M \{|\nabla^2\varphi|^2 - 2Ric(\nabla\varphi, Y)\}dv. \end{aligned} \tag{5.9}$$

Using (5.6), (5.7) and (5.8), we conclude that

$$\begin{aligned}\int_M \{|\nabla X|^2 - |\nabla Y|^2\}dv &= \int_M Ric(\nabla\varphi,\nabla\varphi)dv - n(n-2)\int_M (h^{-1}(R-\rho))^2 dv \\ &= 2n\int_M (h^{-1}(R-\rho))^2 dv - \int_M |\nabla^2\varphi|^2 dv. \end{aligned} \tag{5.10}$$

Combing (5.9) and (5.10), we have the following

$$\int_M |\nabla^2\varphi|^2 dv - n\int_M (h^{-1}(R-\rho))^2 dv - \int_M Ric(\nabla\varphi, Y)dv = 0. \tag{5.11}$$

Plug (5.4) into (5.11), we have

$$\int_M \left|\frac{1}{2}\mathcal{L}_Y g\right|^2 dv = \int_M Ric(\nabla\varphi, Y)dv.$$

Using the assumption, we have $\mathcal{L}_Y g = 0$. Therefore, the proof is complete. □

6 Proof of Theorem 1.4

Since the scalar curvature is constant by assumption, it follows from (2.7) that

$$(n-1)\Delta((R-\rho)u) = -R(R-\rho)u. \tag{6.1}$$

Multiplying (6.1) by $(R-\rho)u$ and integrating it over (M^n,g), we get

$$(n-1)\int_M |\nabla((R-\rho)u)|^2 dv = R\int_M ((R-\rho)u)^2 dv. \tag{6.2}$$

We here assume that the soliton is non-trivial and deduce a contradiction. If M is not trivial, then $(R-\rho)u$ is not a constant function. On the other hand, by (2.5), we have

$$\int_M (R-\rho)u\,dv = 0.$$

Therefore, the first eigenvalue λ_1 of (M^n,g) satisfies

$$\lambda_1 \int_M ((R-\rho)u)^2 dv \le \int_M |\nabla((R-\rho)u)|^2 dv. \tag{6.3}$$

It follows from (6.2) and (6.3) that

$$\lambda_1 \le \frac{R}{n-1},$$

which contradicts (1.13). This completes the proof of Theorem 1.4. □

As a corollary, we have the following:

Corollary 6.1. Any compact gradient $(-\frac{m}{u})$-almost Yamabe soliton with constant negative scalar curvature must be trivial.

From (2.8), it is now easy to deduce the following corollary:

Corollary 6.2. Let $(M^n,g,\nabla u,-\frac{m}{u},\rho)$ be a compact gradient $(-\frac{m}{u})$-almost Yamabe soliton, then $(M^n,g,\nabla u,-\frac{m}{u},\rho)$ is trivial if

$$\int_M Ric(\nabla u,\nabla u)\,dv \le 0.$$

7 Proof of Theorem 1.5

We need the following key lemma:

Lemma 7.1 ([29]). *Let u be an $\mathcal{L}^2$ function, form or tensor that satisfies $\Delta u = \kappa u$ for some $\kappa > 0$. Then u is identically zero.*

Now we prove Theorem 1.5.

Proof of Theorem 1.5. (1) We first observe that

$$\Delta u = -\frac{n}{m}(R-\rho)u.$$

Then, it follows from Lemma 7.1 that $\frac{R-\rho}{m} \geq 0$. On the other hand, from (1.10), we have

$$\nabla_{\nabla u}\nabla u = -\frac{R-\rho}{m}u\nabla u.$$

Therefore, we may use Obata's theorem [30] to obtain that (M^n, g) is isometric to a standard sphere $\mathbb{S}^n\left(\sqrt{\frac{R-\rho}{m}}\right)$.

(2) From (1.10), ∇u defines a closed conformal vector field on (M^n, g). Assume that (M^n, g) is compact with constant scalar curvature R. Hence, from Lemma 5.1, we have that R is positive. Moreover, since (M^n, g) is compact, there exists a point $p \in M$ such that $\nabla u(p) = 0$. Therefore, we may use Lemma 5.2 to obtain that (M^n, g) is isometric to an Euclidean sphere $\mathbb{S}^n$, which concludes the proof of the theorem. □

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