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Non-Negative Integer Matrix Representations of a $\mathbb{Z}_+$-ring

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Abstract. The $\mathbb{Z}_+$-ring is an important invariant in the theory of tensor category. In this paper, by using matrix method, we describe all irreducible $\mathbb{Z}_+$-modules over a $\mathbb{Z}_+$-ring $\mathcal{A}$, where $\mathcal{A}$ is a commutative ring with a $\mathbb{Z}_+$-basis $\{1, x, y, xy\}$ and relations:

$$x^2=1,\ \ y^2=1+x+xy.$$

We prove that when the rank of $\mathbb{Z}_+$-module $n\geq 5$, there does not exist irreducible $\mathbb{Z}_+$-modules and when the rank $n\leq 4$, there exists finite inequivalent irreducible $\mathbb{Z}_+$-modules, the number of which is respectively 1, 3, 3, 2 when the rank runs from 1 to 4.

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1 Introduction

Tensor categories are usually thought as counterparts of groups and rings in the world of categories. They are ubiquitous in noncommutative algebra and representation theory. The $\mathbb{Z}_+$-ring is an important invariant in the theory of tensor category. The terminology on $\mathbb{Z}_+$-rings comes from the paper by Lusztig [1] as well as [2, 3]. Such rings were also studied by Davydov in [4, 5]. The basic concepts and facts about $\mathbb{Z}_+$-rings can be found in [3, 6]. Examples of $\mathbb{Z}_+$-rings include the Green rings of Hopf algebras [7, 8, 9, 10, 11, 12] and the Grothendieck rings of tensor categories [13, 14, 15, 16, 17]. The terminology on $\mathbb{Z}_+$-modules (or $\mathbb{Z}_+$-representations) is from [2, 3, 6] which is different from the

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common representations over rings or algebras. A $\mathbb{Z}_+$-module M is called irreducible if any $\mathbb{Z}_+$-submodule of M is 0 or M. A $\mathbb{Z}_+$-module M is called indecomposable if it is not equivalent to a direct sum of two nonzero $\mathbb{Z}_+$-modules. A $\mathbb{Z}_+$-module over a based ring satisfying the rigid condition is called a based module in [3] or a NIM-representation (the non-negative integer matrix representation) in [18, 19, 20, 21]. It is quite difficult to classify all irreducible $\mathbb{Z}_+$-modules but the job is meaningful and interesting.

In [3], Ostrik proved that for a given $\mathbb{Z}_+$-ring of finite rank there exist only finite inequivalent irreducible $\mathbb{Z}_+$-modules. In the proofs, Ostrik indicated the rank of any irreducible $\mathbb{Z}_+$-module has an upper bound. However, for many situations, the upper bound seems quite large. For a given $\mathbb{Z}_+$-ring, it is difficult to determine whether the rank of an irreducible $\mathbb{Z}_+$-module is less than or equal to the rank of this ring. In this paper, by using matrix method, we describe all irreducible inequivalent $\mathbb{Z}_+$-modules over a $\mathbb{Z}_+$-ring $\mathcal{A}$, where $\mathcal{A}$ is a commutative $\mathbb{Z}_+$-ring with a $\mathbb{Z}_+$-basis $\{1, x, y, xy\}$ and relations: $x^2=1$, $y^2=1+x+xy$. In addition, we find the rank of each irreducible $\mathbb{Z}_+$-module is less than or equal to the rank of $\mathcal{A}$. In fact, the problem is equivalent to studying the irreducible NIM solutions to a system of matrix equations:

$$\begin{cases} A^2=E, \\ B^2-A-E-AB=0, \\ AB=BA. \end{cases} \tag{$*$}$$

We analyze all the situations when the order of the matrix is n, and we get the results completely. When the rank of $\mathbb{Z}_+$-modules $n\geq 5$, there does not exist irreducible $\mathbb{Z}_+$-modules and when the rank $n\leq 4$, there exists finite inequivalent irreducible $\mathbb{Z}_+$-modules, the number of which is respectively 1, 3, 3, 2 when the rank runs from 1 to 4.

The rest of the paper is outlined as follows. In Section 2, we recall some relevant theorems and prove the preliminary propositions we will use in the next sections. In Section 3, we first exhibit the system of matrix equations, then we prove that there does not exist irreducible NIM solutions when order $n\geq 5$. In Section 4, we exhibit the concrete irreducible NIM solutions when order $n\leq 4$. More explicitly, the irreducible $\mathbb{Z}_+$-module over this $\mathbb{Z}_+$-ring $\mathcal{A}$ exists if and only if the rank of module is less than or equal to 4. The concrete irreducible NIM representations can be seen in table 4.1. Furthermore, in each case, we classify the irreducible $\mathbb{Z}_+$-modules under the equivalence in table 4.2.

2 Preliminaries

2.1 Basic definitions and notation

2.1.1 The theory of matrix

We assume all matrices in this paper belong to $\mathbb{M}_n(\mathbb{N})$, where $\mathbb{M}_n(\mathbb{N})$ means the set consisting of n-order square matrices with only natural number elements. For any matrix A, we denote the element at the i-th row and j-th column of A by a_{ij}. If $a_{ij}>0$ for all i,

j, then A is called *positive*. If $a_{ij} \geqslant 0$ for all i, j, then A is called *non-negative*. Besides, if $A \in \mathbb{M}_n(\mathbb{N})$, we call it *non-negative integer matrix* (NIM).

Definition 2.1. Let A be a square matrix with order n. A is called reducible if there exists an n-order permutation matrix P such that $PAP^T = \begin{bmatrix} B & O \\ C & D \end{bmatrix}$, where B is a square matrix with order k and D is a square matrix with order $n-k$ (The definition 2.1 can be found in Gantmachers book [22, Chapter XIII, 1]).

Definition 2.2. Let A and B be solutions of a system of matrix equations $(*)$.
(1) The solutions are called reducible if A and B are reducible simultaneously, namely there exists an n-order permutation matrix P such that

$$PAP^T = \begin{bmatrix} A_1 & O \\ A_2 & A_3 \end{bmatrix}, \ PBP^T = \begin{bmatrix} B_1 & O \\ B_2 & B_3 \end{bmatrix},$$

where A_1 and B_1 are both square matrices with the same order, A_3 and B_3 are both square matrices with the same order.
(2) The solutions are called irreducible if A and B are irreducible simultaneously, namely there doesn't exist an n-order permutation matrix P such that

$$PAP^T = \begin{bmatrix} A_1 & O \\ A_2 & A_3 \end{bmatrix}, \ PBP^T = \begin{bmatrix} B_1 & O \\ B_2 & B_3 \end{bmatrix},$$

where A_1 and B_1 are both square matrices with the same order, A_3 and B_3 are both square matrices with the same order.

2.1.2 $\mathbb{Z}_+$-ring and $\mathbb{Z}_+$-module

Definition 2.3. Let $\mathcal{A}$ be a ring which is free as a $\mathbb{Z}$-module.
(1) A $\mathbb{Z}_+$-basis of $\mathcal{A}$ is a $\mathbb{Z}$-basis $\mathcal{B} = \{b_i\}_{i\in I}$ such that $b_i b_j = \sum_{k\in I} c_{ij}^k b_k$, where $c_{ij}^k \in \mathbb{Z}_+$.
(2) A $\mathbb{Z}_+$-ring is a ring with a xed $\mathbb{Z}_+$-basis and unit 1 which is a non-negative linear combination of the basis elements.

Definition 2.4. Let $\mathcal{A}$ be a $\mathbb{Z}_+$-ring with the basis $\{b_i\}_{i\in I}$. A $\mathbb{Z}_+$-module over $\mathcal{A}$ is an $\mathcal{A}$-module M with a fixed $\mathbb{Z}$-basis $\{m_l\}_{l\in J}$ such that all the structure constants a_{il}^k defined by the equality $b_i m_l = \sum_k a_{il}^k m_k$ are non-negative integers.

A $\mathbb{Z}_+$-module can be alternatively defined as follows:

Definition 2.5. Let $\mathcal{A}$ be a $\mathbb{Z}_+$-ring with the basis $\{b_i\}_{i\in I}$ such that $b_i b_j = \sum_{k\in I} c_{ij}^k b_k$. For a $\mathbb{Z}_+$-module M over $\mathcal{A}$, we also mean an assignment of each b_i to a matrix M_i with non-negative integer entries such that M forms a representation of $\mathcal{A}$: $M_i M_j = \sum_{k\in I} c_{ij}^k M_k$, for all $i,j,k \in I$, and also that the identity of $\mathcal{A}$ is assigned to the identity matrix. The rank of a $\mathbb{Z}_+$-module M equals to the size of the matrices M_i.

Definition 2.6. We call two $\mathbb{Z}_+$-modules M, M' over $\mathcal{A}$ with bases $\{m_i\}_{i\in J}$ and $\left\{m'_j\right\}_{j\in J'}$ equivalent, if there exists a bijection $\varphi: J \to J'$ such that the induced $\mathbb{Z}$-linear map $\tilde{\varphi}$ of abelian groups M, M' defined by $\tilde{\varphi}(m_i) = m'_{\varphi(i)}$ is an isomorphism of $\mathcal{A}$-modules. In other words, we call two $\mathbb{Z}_+$-module M and M' of rank n equivalent if there exists an nn permutation matrix P such that $M'_i = PM_iP^{-1}$ for all $i \in I$.

Definition 2.7. For a $\mathbb{Z}_+$-submodule N of a $\mathbb{Z}_+$-module M with a basis $\{m_l\}_{l\in J}$, we mean an abelian subgroup of M spaned by $\{m_l\}_{l\in J'}$, where J' is a subset of J such that N is an $\mathcal{A}$-submodule.

Definition 2.8. We call a $\mathbb{Z}_+$-module M irreducible if any $\mathbb{Z}_+$-submodule of M is 0 or M (In other words, the $\mathbb{Z}$-span of any proper subset of the basis of M is not an $\mathcal{A}$-submodule).

The irreducible $\mathbb{Z}_+$-module can also be defined as follows:

Definition 2.9. A $\mathbb{Z}_+$-module M is called reducible if M_i is simultaneously reducible for any $i \in I$. M is called irreducible when all M_i cannot be simultaneously put into triangular block form and it is called irreducible NIM-representation.

2.2 Auxiliary results

To prove our results we will need the following statements.

Theorem 2.1. *The necessary and sufficient condition for the non-negative matrix A with order n to be irreducible is $(E+A)^{n-1} > 0$.*

Proof. A proof of the theorem above can be found in Gantmachers book [22, Chapter XIII, 1]. □

Theorem 2.2. *Let $A = (a_{ij})_{n\times n}$ be a non-negative irreducible matrix. Then there exists an eigenvalue λ of A and λ is a real number which satisfies $\lambda > 0$. λ is a simple root such that $\lambda \geq |\alpha|$, where α is any eigenvalue of A. We call the eigenvalue λ Perron-Frobenius eigenvalue.*

Proof. A proof of the theorem above can be found in Gantmachers book [22, Chapter XIII, 2]. □

2.3 Preliminary propositions

Proposition 2.1. If $A, B \in \mathbb{M}_n(\mathbb{N})$ and satisfy $AB = E$. Then A and B are both permutation matrices.

Proof. Let $A = (a_{ij})_{n\times n}$ be a matrix with order n where $a_{ij} \in \mathbb{N}$ and $B = (b_{ij})_{n\times n}$ be a matrix with order n where $b_{ij} \in \mathbb{N}$. Under the circumstance, we claim that any element of A

and B is 0 or 1. We can assume that there exists an element of A which satisfies $a_{ij} \geq 2$. Consider the element of E,

$$\delta_{ij} = \begin{cases} 1(i=j) \\ 0(i \neq j) \end{cases}, \quad \delta_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ij}b_{jj} + \cdots + a_{in}b_{nj},$$

where $a_{i1}b_{1j} \geq 0$, $a_{i2}b_{2j} \geq 0, \ldots, a_{ij}b_{jj} \geq 2b_{jj}, \ldots, a_{in}b_{nj} \geq 0$. In this case, we can get $b_{jj} = 0$. Similarly, considering the elements $\delta_{i1}\delta_{i2}\cdots\delta_{in}$, we can get $b_{j1}, b_{j2}\cdots b_{jn} = 0$. But under this circumstance, there exists a line where all elements are 0 in B which contradicts with the fact that the determinant of B is not 0. So, the assumption does not holds, and any element of A is 0 or 1. According to $AB = E$, we can get $BA = E$. By using the same method, we can prove any element of B is 0 or 1 as well. Now we will prove that there is only one 1 in per row and per column of A and B.

Since A and B are both reversible, there exists at least one 1 in per row and per column. We give the following discussion to prove that there exists at most one 1 in per row and per column. We might as well assume a_{i1}, a_{i2} are both 1 and the remaining elements of the i-th row are 0. Considering the element δ_{ii} of E, $\delta_{ii} = a_{i1}b_{1i} + a_{i2}b_{2i} + \cdots + a_{in}b_{ni} = 1$ and we have $b_{1i} = 0$ or $b_{2i} = 0$. We might as well assume that $b_{1i} = 0$, then $b_{2i} = 1$. Considering that the elements of E, $\delta_{i1}, \delta_{i2}, \cdots, \delta_{ii-1}, \delta_{ii+1}, \cdots, \delta_{in}$ are all equal to 0, we have that $b_{11} = 0$, $b_{12} = 0, \cdots$, $b_{1i-1} = 0$, $b_{1i+1} = 0, \cdots$, $b_{1n} = 0$. Combined with $b_{1i} = 0$ we know there exists a zero line in B which contradicts with the fact that the determinant of B is not equal to 0. Therefore, we have the conclusion that there exists at most one 1 in per row and per column of A and the same conclusion is true on matrix B. Hence, there is only one 1 in per row and per column of A and B. Then A and B are both permutation matrices. □

Corollary 2.3. If $A \in \mathbb{M}_n(\mathbb{N})$ and satisfies $A^k = E$, $k \geq 1$, then A must be a permutation matrix.

Proof. Let $B = A^{k-1}$ and we have $AB = E$ according to $A^k = E$, then by proposition 2.1 we get A must be a permutation matrix. □

3 The irreducible NIM representations when $n \geq 5$

In this section, we will prove there does not exist irreducible $\mathbb{Z}_+$-module M when the rank of M satisfies $n \geq 5$ over the ring $\mathcal{A}$, where $\mathcal{A}$ is a commutative ring with a $\mathbb{Z}_+$-basis $\{1, x, y, xy\}$ and relations: $x^2 = 1$, $y^2 = 1 + x + xy$. This question is equivalent to proving that there does not exist irreducible NIM solutions to the following equations:

$$\begin{cases} A^2 = E, \\ B^2 - A - E - AB = 0, \\ AB = BA, \end{cases} \tag{$*$}$$

where A and $B \in \mathbb{M}_n(\mathbb{N})$ are NIM solutions to $(*)$. Now in the following, we let $b = E + A + B + AB$. Then it is easy to see that $b \in \mathbb{M}_n(\mathbb{N})$.

Theorem 3.1. *If A and B are irreducible NIM solutions, then b is irreducible.*

Proof. We assume b is reducible, then there exists a permutation matrix P such that:

$$PbP^T = P(E+A+B+AB)P^T = E+PAP^T+PBP^T+PABP^T = \begin{bmatrix} b_1 & O \\ b_2 & b_3 \end{bmatrix}.$$

Due to the fact that A, B and AB are all non-negative integer matrices, we have the following results:

$$PAP^T = \begin{bmatrix} A_1 & O \\ A_2 & A_3 \end{bmatrix}, \ PBP^T = \begin{bmatrix} B_1 & O \\ B_2 & B_3 \end{bmatrix}, \ PABP^T = \begin{bmatrix} C_1 & O \\ C_2 & C_3 \end{bmatrix}.$$

In this case, A and B must be reducible solutions which contradicts with the assumption. Therefore, we claim b is irreducible. □

Theorem 3.2. *If A and B are irreducible NIM solutions, then* $b>0$.

Proof. We already know that b is an irreducible non-negative matrix by theorem 3.1, so we can get $(E+b)^{n-1}>0$ by theorem 2.1. According to the binomial expansion and the system of matrix equations $(*)$, we have

$$(E+b)^{n-1} = m_1E+m_2A+m_3B+m_4AB>0, \ m_1,m_2,m_3,m_4 \in \mathbb{Z}_+.$$

Now we claim $b=E+A+B+AB>0$. Otherwise, if there exists one element b_{ij} of matrix b such that $b_{ij}=0$, due to the fact that A, B and AB are all non-negative integer matrices, the elements of the i-th row and the j-th column for each matrix E, A, B and AB are all 0, which contradicts with the fact that $m_1E+m_2A+m_3B+m_4AB>0$. Thereby we can get $b=E+A+B+AB>0$. □

Theorem 3.3. *If* $b>0$*, then b is irreducible and* $\mathrm{tr}(b)=6$.

Proof. Firstly, the irreducibility of b is from Theorem 2.1. According to the system of matrix equations $(*)$, we have $b^2=6b$. Assume λ is is the eigenvalue of b. Then we have $\lambda^2=6\lambda$, $\lambda=6$ or 0. Then we claim there must exist one eigenvalue which equals to 6. Otherwise the trace of b satisfies $\mathrm{tr}(b)=0$, and it contradicts with $b>0$, $\mathrm{tr}(b)>0$. Considering that b is irreducible and $b>0$, by Theorem 2.2 we get 6 is the only non-zero single root of b. Therefore, we can get $\mathrm{tr}(b)=6$. □

Theorem 3.4. *Let A and B in* $\mathbb{M}_n(\mathbb{N})$ *when* $n\geq 7$ *be solutions to the system of matrix equations* $(*)$*. Then A and B must be reducible solutions.*

Proof. Assume there exist irreducible NIM solutions A and B. We have known $b=E+A+B+AB$ and by Theorem 3.2 we get $b>0$. Therefore $\mathrm{tr}(b)\geq \mathrm{tr}(E)=n\geq 7$ which contradicts with the fact $\mathrm{tr}(b)=6$ by Theorem 3.3. In a word, the assumption is incorrect, so the NIM solutions are always reducible. □

Theorem 3.5. *If $n=5$, there is no irreducible NIM solution to the system of matrix equations $(*)$.*

Proof. We assume there exist irreducible NIM solutions A and B, then by Theorem 3.3 we know that $\mathrm{tr}(b)=6$. According to $b>0$, there are several cases about the values of the diagonal elements of b. We might as well let the diagonal elements of b respectively be $b_{11}=2$, $b_{ii}=1$ (i=2,3,4,5) and other cases are similar. We notice A is a permutation matrix according to $A^2=E$ by corollary 2.1, and the order of A is odd, so there is at least one 1 in the diagonal elements of A and $a_{11}=1$. Besides, since $\mathrm{tr}(b)=\mathrm{tr}(E+A+B+AB)$, $\mathrm{tr}(A)\geq 1$ and $\mathrm{tr}(E)=5$, we have $\mathrm{tr}(AB)=0$, $\mathrm{tr}(B)=0$, and $\mathrm{tr}(A)=1$. Considering A is NIM and satisfies $A^2=E$, we get A is one of the following three matrices:

$$(1)\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad (2)\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \quad (3)\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

Notation: We only discuss one situation when $A=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$ because other situations about the choice of A are similar.

According to $\mathrm{tr}(AB)=0$ and $\mathrm{tr}(B)=0$, we get B must be the following matrix:

$$B=\begin{bmatrix} 0 & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & 0 & 0 & a_{24} & a_{25} \\ a_{31} & 0 & 0 & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & 0 & 0 \\ a_{51} & a_{52} & a_{53} & 0 & 0 \end{bmatrix}.$$

Since $AB=BA$, we know that $a_{12}=a_{13}, a_{14}=a_{15}, a_{21}=a_{31}, a_{24}=a_{35}, a_{25}=a_{34}, a_{41}=a_{51}, a_{42}=a_{53}$, $a_{52}=a_{43}$. We have the system of diagonal element equations as follows by $B^2=A+E+AB$:

$$\begin{cases} 2a_{12}a_{21}+2a_{14}a_{41}=2, \\ a_{12}a_{21}+a_{24}a_{42}+a_{25}a_{43}=1, \\ a_{14}a_{41}+a_{24}a_{42}+a_{25}a_{43}=1, \end{cases}$$

so we get $2a_{14}a_{41}=1$, which contradicts with the fact that a_{14} and a_{41} are non-negative integers. It is obvious that the rest situations are similar to the above and we can get contradictions as well. Therefore, the assumption doesn't hold, that is to say when $n=5$ if the solutions A and B of the matrix equations exist, they must be reducible. In other words, there doesn't exist irreducible $\mathbb{Z}_+$-modules when the rank of the module is 5. □

Theorem 3.6. *If $n=6$, there is no NIM irreducible solution to the system of matrix equations $(*)$.*

Proof. We assume that there exists irreducible NIM solutions A and B. By Theorem 3.3, we get $\mathrm{tr}(b)=6$. Since $b>0$, we get $b=\left(b_{ij}\right)_{6\times 6}$, where $b_{ii}=1(i=1,2,3,4,5,6)$. Its easy to know that all of the diagonal elements of A, AB and B equal to 0. Then according to corollary 2.1, we get A is one of the following fifteen matrices:

$$(1)\begin{bmatrix}0&1&0&0&0&0\\1&0&0&0&0&0\\0&0&0&1&0&0\\0&0&1&0&0&0\\0&0&0&0&0&1\\0&0&0&0&1&0\end{bmatrix},\quad (2)\begin{bmatrix}0&1&0&0&0&0\\1&0&0&0&0&0\\0&0&0&0&0&1\\0&0&0&0&1&0\\0&0&0&1&0&0\\0&0&1&0&0&0\end{bmatrix},\quad (3)\begin{bmatrix}0&1&0&0&0&0\\1&0&0&0&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\\0&0&1&0&0&0\\0&0&0&1&0&0\end{bmatrix},$$

$$(4)\begin{bmatrix}0&0&0&1&0&0\\0&0&0&0&0&1\\0&0&0&0&1&0\\1&0&0&0&0&0\\0&0&1&0&0&0\\0&1&0&0&0&0\end{bmatrix},\quad (5)\begin{bmatrix}0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\\1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\end{bmatrix},\quad (6)\begin{bmatrix}0&0&0&0&0&1\\0&0&0&0&1&0\\0&0&0&1&0&0\\0&0&1&0&0&0\\0&1&0&0&0&0\\1&0&0&0&0&0\end{bmatrix},$$

$$(7)\begin{bmatrix}0&0&0&0&0&1\\0&0&1&0&0&0\\0&1&0&0&0&0\\0&0&0&0&1&0\\0&0&0&1&0&0\\1&0&0&0&0&0\end{bmatrix},\quad (8)\begin{bmatrix}0&0&0&0&0&1\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\1&0&0&0&0&0\end{bmatrix},\quad (9)\begin{bmatrix}0&0&0&0&1&0\\0&0&0&0&0&1\\0&0&0&1&0&0\\0&0&1&0&0&0\\1&0&0&0&0&0\\0&1&0&0&0&0\end{bmatrix},$$

$$(10)\begin{bmatrix}0&0&1&0&0&0\\0&0&0&0&1&0\\1&0&0&0&0&0\\0&0&0&0&0&1\\0&1&0&0&0&0\\0&0&0&1&0&0\end{bmatrix},\quad (11)\begin{bmatrix}0&0&1&0&0&0\\0&0&0&0&0&1\\1&0&0&0&0&0\\0&0&0&0&1&0\\0&0&0&1&0&0\\0&1&0&0&0&0\end{bmatrix},\quad (12)\begin{bmatrix}0&0&0&1&0&0\\0&0&1&0&0&0\\0&1&0&0&0&0\\1&0&0&0&0&0\\0&0&0&0&0&1\\0&0&0&0&1&0\end{bmatrix},$$

$$(13)\begin{bmatrix}0&0&1&0&0&0\\0&0&0&1&0&0\\1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&0&0&0&1\\0&0&0&0&1&0\end{bmatrix},\quad (14)\begin{bmatrix}0&0&0&0&1&0\\0&0&1&0&0&0\\0&1&0&0&0&0\\0&0&0&0&0&1\\1&0&0&0&0&0\\0&0&0&1&0&0\end{bmatrix},\quad (15)\begin{bmatrix}0&0&0&0&1&0\\0&0&0&1&0&0\\0&0&0&0&0&1\\0&1&0&0&0&0\\1&0&0&0&0&0\\0&0&1&0&0&0\end{bmatrix}.$$

Notation: We only discuss the situation when $A=\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$ because other situations about the choice of A are similar.

According to $\mathrm{tr}(AB)=0$, $\mathrm{tr}(B)=0$, we get B must be the following:

$$B=\begin{bmatrix} 0 & 0 & a_{13} & a_{14} & a_{15} & a_{16} \\ 0 & 0 & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & 0 & 0 & a_{35} & a_{36} \\ a_{41} & a_{42} & 0 & 0 & a_{45} & a_{46} \\ a_{51} & a_{52} & a_{53} & a_{54} & 0 & 0 \\ a_{61} & a_{62} & a_{63} & a_{64} & 0 & 0 \end{bmatrix}.$$

Since we have $AB=BA$, we get $a_{13}=a_{24}, a_{14}=a_{23}, a_{15}=a_{26}, a_{16}=a_{25}, a_{31}=a_{42}, a_{32}=a_{41}, a_{35}=a_{46}$, $a_{36}=a_{45}$, $a_{51}=a_{62}$, $a_{52}=a_{61}$, $a_{53}=a_{64}$, $a_{54}=a_{63}$. According to $B^2=A+E+AB$ we get the system of diagonal element equations as follows:

$$\begin{cases} a_{13}a_{31}+a_{14}a_{32}+a_{15}a_{51}+a_{16}a_{52}=1, \\ a_{13}a_{31}+a_{32}a_{14}+a_{35}a_{53}+a_{36}a_{54}=1, \\ a_{15}a_{51}+a_{16}a_{52}+a_{35}a_{53}+a_{36}a_{54}=1. \end{cases}$$

Hence we get $2(a_{13}a_{31}+a_{14}a_{32})=1$, which contradicts with the fact that $a_{13}, a_{31}, a_{14}, a_{32}$ are non-negative integers. Therefore, the solution doesn't exist. The rest situations are the same as above and we can get contradictions as well. Therefore, the assumption doesn't hold, that is to say if NIM solutions A and B of the matrix equations $(*)$ exist, they must be reducible when the order is 6. In other words, there doesn't exist irreducible $\mathbb{Z}_+$-modules when the rank of the module is 6. □

4 The irreducible NIM representations when $n\leq 4$

We have proved that there do not exist irreducible NIM solutions when order $n\geq 5$. In this section, we will calculate all the irreducible NIM solutions to matrix equations $(*)$ when $n\leq 4$.

• When $n=1$, there is only one irreducible NIM solution:

$$A=1,\ B=2.$$

• When $n=2$, according to the fact that A^2=E and $A\in\mathbb{M}_n(\mathbb{N})$, we have A is one of the following two matrices:

$$(1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad (2)\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Case (1): Let $A=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B=\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$. Since $AB=BA=B$, we get $B^2=B+2E$, so we have the system of equations as follows:

$$b_{11}^2+b_{12}b_{21}=b_{11}+2, \tag{4.1}$$
$$b_{11}b_{12}+b_{12}b_{22}=b_{12}, \tag{4.2}$$
$$b_{21}b_{11}+b_{22}b_{21}=b_{21}, \tag{4.3}$$
$$b_{21}b_{12}+b_{22}^2=b_{22}+2. \tag{4.4}$$

We know $b_{11}\le 2$ with (4.1). Now we will discuss about the value of b_{11}.

Case (1.1): If $b_{11}=2$, $b_{12}b_{21}=0$, we get $b_{12}(1+b_{22})=0$ with (4.2) and $b_{21}(1+b_{22})=0$ with (4.3), so we have $b_{12}=b_{21}=0$. Besides, we know $b_{22}=2$ by $b_{22}^2=b_{22}+2$. Then

$$B=B_1=\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$$

Case (1.2): If $b_{11}=1$, $b_{12}b_{21}=2$, we get $b_{12}b_{22}=0$, $b_{22}=0$ with (4.2) and $b_{22}b_{21}=0$ with (4.3). Therefore, we have $b_{12}=2, b_{21}=1$ or $b_{12}=1, b_{21}=2$. Then

$$B=B_2=\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \quad \text{or} \quad B=B_3=\begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}.$$

Case (1.3): If $b_{11}=0$, $b_{12}b_{21}=2$, we get $b_{12}b_{22}=b_{12}$ with (4.2) , $b_{22}b_{21}=b_{21}$ with (4.3) and $b_{22}^2=b_{22}$ with (4.4). That is to say b_{22} = 0 or 1. Therefore, if $b_{22}=0$, we get $b_{12}=b_{21}=0$ and then

$$B=B_4=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

If $b_{22}=1$, we get $b_{12}=2, b_{21}=1$ or $b_{12}=1, b_{21}=2$ and then

$$B=B_5=\begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} \quad \text{or} \quad B=B_6=\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}.$$

Case (2): Let $A=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. According to $AB=BA$, we get $b_{12}=b_{21}$, $b_{11}=b_{22}$, then we have the system of equations as follows by $B^2=E+A+AB$:

$$b_{11}^2+b_{12}b_{21}=b_{11}+1, \tag{4.5}$$
$$b_{11}b_{12}+b_{12}b_{22}=b_{22}+1, \tag{4.6}$$
$$b_{21}b_{11}+b_{22}b_{21}=b_{11}+1, \tag{4.7}$$
$$b_{21}b_{12}+b_{22}^2=b_{12}+1. \tag{4.8}$$

With (4.5), we get $b_{11}^2+b_{22}^2=b_{12}+1$, so we have $b_{11}\leq 1$.

Case (2.1): If $b_{11}=1$, we get $b_{12}b_{21}=1$ and $b_{12}=b_{21}=1$. Then we have $b_{22}^2=1$ and $b_{22}=1$. Therefore, we have

$$B=B_7=\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Case (2.2): If $b_{11}=b_{22}=0$, we will get $b_{12}^2=b_{12}+1$ with (4.5), but there is no integer solution, so this case is impossible.

• When $n=3$, according to $A^2=E$ and $A\in\mathbb{M}_n(\mathbb{N})$, we know that A is one of the following four matrices:

$$(1)\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\quad (2)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\quad (3)\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\quad (4)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

By the same analyzing and calculating method as the case when $n=2$, we can easily calculate all the irreducible NIM solutions to $(*)$. Therefore, the calculating process is omitted and the results are shown in Table 1.

• When $n=4$, since $A^2=E$ and $A\in\mathbb{M}_n(\mathbb{N})$, we know that A is one of the following ten matrices:

$$(1)\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},\ (2)\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},\ (3)\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},\ (4)\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$(5)\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix},\ (6)\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},\ (7)\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},\ (8)\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$$(9)\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},\ (10)\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

By the same analyzing and calculating method as the case when $n=2$, we can easily calculate all the irreducible NIM solutions to $(*)$. Therefore, the calculating process is omitted and the results are shown in Table 1.

According to the discussion above, we calculate all the irreducible NIM solutions to the matrix equations $(*)$ when $n\leq 4$ and get the following theorem.

Theorem 4.1. *All the irreducible NIM solutions to* $(*)$ *are as follows:*

Table 1:

All the irreducible NIM solutions to $(*)$

n	A	B
1	1	2
2	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$
	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$
3	$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$
	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$
	$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$
	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$
4	$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix},$ $\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}.$
	$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix},$ $\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}.$
	$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix},$ $\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$

Now, according to Table 1, we classify all the irreducible $\mathbb{Z}_+$-modules under the equivalence by Definition 2.6, and we have the following theorem.

Theorem 4.2. *All the inequivalent irreducible NIM solutions to* $(*)$ *are as follows:*

Table 2:

All the inequivalent irreducible NIM solutions to $(*)$

n	A	B
1	1	2
2	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$
	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$
3	$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$
	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$
4	$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$

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