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On Doubly Twisted Product of Complex Finsler Manifolds

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Abstract. Let (M_1,F_1) and (M_2,F_2) be two strongly pseudoconvex complex Finsler manifolds. The doubly twisted product (abbreviated as DTP) complex Finsler manifold $(M_1\times_{(\lambda_1,\lambda_2)}M_2,F)$ is the product manifold $M_1\times M_2$ endowed with the twisted product complex Finsler metric $F^2=\lambda_1^2F_1^2+\lambda_2^2F_2^2$, where λ_1 and λ_2 are positive smooth functions on $M_1\times M_2$. In this paper, the relationships between the geometric objects (e.g. complex Finsler connections, holomorphic and Ricci scalar curvatures, and real geodesic) of a DTP-complex Finsler manifold and its components are derived. The necessary and sufficient conditions under which the DTP-complex Finsler manifold is a Kähler Finsler (respctively weakly Kähler Finsler, complex Berwald, weakly complex Berwald, complex Landsberg) manifold are obtained. By means of these, we provide a possible way to construct a weakly complex Berwald manifold, and then give a characterization for a complex Landsberg metric that is not a Berwald metric.

AMS subject classifications: 53C60, 53C40

Key words: Doubly twisted products, complex Finsler metric, holomorphic curvature, geodesic

1 Introduction

Warped product and twisted product are important methods used to produce new class of geometrical spaces, and these products are widely applied in theoretical physics. The notion of warped product of two Riemannian manifolds was first introduced by O'Neill and Bishop to construct Riemannian manifolds with negative curvature [11], then it was studied by many authors [10,15,16,23]. Asanov considered the warped product of Finsler

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manifolds and obtained some models of relativity theory [3, 4]. Kozma, Peter and Varga extended the notion of warped product to the real Finsler manifolds [19]. Many properties of such kinds of products were studied (see [9, 19, 24, 35, 38]). In 2016, He and Zhong systematically studied the doubly warped product of complex Finsler manifolds [14], they got a new method of constructing weakly complex Berwald metric through the doubly warped product of complex Finsler metric.

The notion of twisted product of Riemannian manifolds was mentioned first by Chen in [13], and was generalized for the pseudo-Riemannian case by Ponge and Reckziegel [26]. Kozma, Peter, and Shimada extended the construction of twisted product to real Finslerian case [18], they presented the construction of twisted product of Finsler manifolds and investigate some geometrical properties relating to Cartan connection, geodesics and completeness. In [25], Peyghan, Tayebi and Nourmohammadi obtained the Riemannian curvature and some of non-Riemannian curvatures of the twisted product Finsler manifold such as Berwald curvature, mean Berwald curvature, and studied locally dually flat twisted product Finsler manifold. In [34], Wang gave the necessary and sufficient conditions for multiply twisted product Finsler manifolds to be Riemannian, Landsberg, Berwald, locally dually flat and locally Minkowski.

Recently, Zhong proved that there are lots of strongly pseudoconvex (even strongly convex) unitary invariant complex Finsler metrics in domains in $\mathbb{C}^n$ [37]. The purpose of this paper is to study the doubly twisted product of strongly pseudoconvex complex Finsler manifolds.

This paper is organized as follows. In Section 2, we recall some basic concepts and notations of complex Finsler geometry and extend the doubly twisted product to complex Finsler manifold. In Section 3, we deduce the most commonly used complex Finsler connections (the Chern-Finsler connection, the complex Rund connection, the complex Berwald connection, and the complex Hashiguchi connection, etc.) of the DTP-complex Finsler manifold, which are expressed by the corresponding connections of its components. In Section 4, we derive the formulae of the holomorphic curvature and Ricci scalar curvature of the DTP-complex Finsler manifold, which are expressed by the holomorphic and Ricci scalar curvatures of its components. In Section 5, we derive the real geodesic equations of the DTP-complex Finsler manifold. In Section 6, we obtain the necessary and sufficient conditions under which the DTP-complex Finsler manifold to be Kähler Finsler (respctively weakly Kähler Finsler, complex Berwald, weakly complex Berwald, complex Landsberg, complex locally Minkowski) manifold. By using of these results, we provide a possible way to construct special complex Finsler manifolds.

2 Preliminary

In this section, we recall some basic concepts and notations which will be used in this paper, and give the definition of the DTP-complex Finsler manifolds.

Let M be a complex manifold of complex dimension n, and $T^{1,0}M$ be the holomorphic tangent bundle of M. We denote $z=(z^1,\cdots,z^n)$ the local holomorphic coordinates on M, and $(z,v)=(z^1,\cdots,z^n,v^1,\cdots,v^n)$ the induced local holomorphic coordinates on $T^{1,0}M$. We shall assume that M is endowed with a strongly pseudoconvex complex Finsler metric F in the following sense.

Definition 2.1 ([1]). *A strongly pseudoconvex complex Finsler metric F on a complex manifold M is a continuous function $F:T^{1,0}M\to\mathbb{R}^+$ satisfying*

(i) $G=F^2$ *is smooth on* $\widetilde{M}=T^{1,0}M-\{\text{zero section}\}$;

(ii) $F(p,v)>0$ *for all* $(p,v)\in\widetilde{M}$;

(iii) $F(p,\zeta v)=|\zeta|F(p,v)$ *for all* $(p,v)\in T^{1,0}M$ *and* $\zeta\in\mathbb{C}$;

(iv) *the Levi matrix (or complex Hessian matrix)*

$$(G_{\alpha\overline{\beta}})=\left(\frac{\partial^2 G}{\partial v^\alpha\partial\overline{v^\beta}}\right) \tag{2.1}$$

is positive definite on $\tilde{M}$.

In this paper, let $(G^{\bar{\nu}\beta})$ be the inverse matrix of $(G_{\alpha\bar{\nu}})$ such that $G^{\bar{\nu}\beta}G_{\alpha\bar{\nu}}=\delta_\alpha^\beta$. We also use the notion in [1], that is, we shall adopt the semicolon to distinguish between the derivatives of G with respect to the v-coordinates and z-coordinates, for example:

$$G_{\bar{\mu};\nu}=\frac{\partial^2 G}{\partial z^\nu\partial\overline{v^\mu}},\qquad G_{\alpha;\bar{\nu}}=\frac{\partial^2 G}{\partial\overline{z^\nu}\partial v^\alpha}.$$

The property of G is its (1,1)-homogeneity in the sense that

$$G(z,\lambda v)=\lambda\bar{\lambda}G(z,v), \tag{2.2}$$

for all $(z,v)\in T^{1,0}M$ and $\lambda\in C^*$.

As the consequence of (2.2), we get

$$\begin{aligned}
&G_\alpha v^\alpha=G,\quad G_{\alpha\beta}v^\alpha=0,\quad G_{\alpha\beta}v^\alpha v^\beta=0,\quad G_{\alpha\bar{\beta}}v^\alpha=G_{\bar{\beta}},\\
&G_{\bar{\alpha}}\overline{v^\alpha}=G,\quad G_{\bar{\alpha}\bar{\beta}}\overline{v^\alpha}=0,\quad G_{\alpha\bar{\beta}}v^\alpha\overline{v^\beta}=G,\quad G_{\alpha\bar{\beta}}\overline{v^\beta}=G_\alpha,\\
&G_{\alpha\beta\gamma}v^\alpha=-G_{\beta\gamma},\quad G_{\alpha\bar{\beta}\gamma}v^\alpha=0,\quad G_{\alpha\beta\bar{\gamma}}\overline{v^\gamma}=G_{\alpha\beta}.
\end{aligned}$$

Let (M_1,F_1) and (M_2,F_2) be two strongly pseudoconvex complex Finsler manifolds with $\dim_{\mathbb{C}}M_1=m$ and $\dim_{\mathbb{C}}M_2=n$, then $M=M_1\times M_2$ is a strongly pseudoconvex complex Finsler manifold with $\dim_{\mathbb{C}}M=m+n$.

Throughout this paper we use the natural product coordinate system on the product manifold $M_1\times M_2$. Let (p_0,q_0) be a point in M, then there are coordinate chart (U,z_1) and (V,z_2) on M_1 and M_2, respectively, such that $p_0\in U$ and $q_0\in V$. Therefore we get a coordinate chart (W,z) on M such that $W=U\times V$ and $(p_0,q_0)\in W$, and for all $(p,q)\in W$, $z(p,q)=(z_1(p),z_2(q))$, where $z_1=(z^1,\ldots,z^m)$, $z_2=(z^{m+1},\ldots,z^{m+n})$.

Let $T^{1,0}M_1$, $T^{1,0}M_2$ and $T^{1,0}M$ be the holomorphic tangent bundles of M_1, M_2 and M, respectively. Let $\pi_1:M_1\times M_2\to M_1$, $\pi_2:M_1\times M_2\to M_2$ be natural projection maps, then $d\pi_1:T^{1,0}(M_1\times M_2)\to T^{1,0}M_1$, $d\pi_2:T^{1,0}(M_1\times M_2)\to T^{1,0}M_2$ be the holomorphic tangent maps induced by π_1 and π_2, respectively. Note that $d\pi_1(z,v)=(z_1,v_1)$ and $d\pi_2(z,v)=(z_2,v_2)$ for every $v=(v_1,v_2)\in T_z^{1,0}(M_1\times M_2)$ with $v_1=(v^1,\cdots,v^m)\in T_{z_1}^{1,0}M_1$ and $v_2=(v^{m+1},\cdots,v^{m+n})\in T_{z_2}^{1,0}M_2$. Denote $\widetilde{M_1}=T^{1,0}M_1-\{\text{zero section}\}$, $\widetilde{M_2}=T^{1,0}M_2-\{\text{zero section}\}$, $\widetilde{M}=\widetilde{M_1}\times\widetilde{M_2}\subset T^{1,0}(M_1\times M_2)-\{\text{zero section}\}$.

Definition 2.2. *Let (M_1,F_1) and (M_2,F_2) be two strongly pseudoconvex complex Finsler manifolds and $\lambda_i:M_1\times M_2\to(0,+\infty)(i=1,2)$ be smooth functions. The doubly twisted product (abbreviated as DTP) complex Finsler manifold of (M_1,F_1) and (M_2,F_2) is the product complex manifold $M=M_1\times M_2$ endowed with the complex Finsler metric $F:\widetilde{M}\to\mathbb{R}^+$ given by*

$$F^2(z,v)=\lambda_1^2(z)F_1^2(\pi_1(z),\,d\pi_1(v))+\lambda_2^2(z)F_2^2(\pi_2(z),d\pi_2(v)),\tag{2.3}$$

for $z=(z_1,z_2)\in M$ and $v=(v_1,v_2)\in T_z^{1,0}M-\{\text{zero section}\}$. The functions λ_1 and λ_2 are called twisted functions. The DTP-complex Finsler manifold of (M_1,F_1) and (M_2,F_2) is denoted by $(M_1\times_{(\lambda_1,\lambda_2)}M_2,F)$.

It is clearly that the function F defined by (2.3) is a strongly pseudoconvex complex Finsler metric on M. Since F_i is smooth on $T^{1,0}M_i$ if and only if F_i comes from a Hermitian metric on M_i for $i=1,2$, the metric F must to be defined on $\widetilde{M_1}\times\widetilde{M_2}$ rather than on $\widetilde{M}$, or on $\widetilde{M_1}\times T^{1,0}M_2$, or on $T^{1,0}M_1\times\widetilde{M_2}$.

In the case of $\lambda_1\equiv1$, the corresponding DTP-complex Finsler manifold is called twisted product complex Finsler manifold, further, if λ_2 depends only on the point of M_1, then $(M_1\times_{(1,\lambda_2)}M_2,F)$ becomes warped product complex Finsler manifold. If λ_1 and λ_2 depend only on the point of M_2 and M_1, respectively, then we have a doubly warped product complex Finsler manifold. If neither λ_1 nor λ_2 is a constant, then we call $(M_1\times_{(\lambda_1,\lambda_2)}M_2,F)$ a nontrivial (proper) DTP-complex Finsler manifold of (M_1,F_1) and (M_2,F_2).

It also should be mentioned that the conformal transformation of a complex Finsler metric can be interpreted as a twisted product, namely one where the first factor M_1 consist of one point only. Therefore formulae and assertions for DTP-complex Finsler manifold are applied in many situations.

Notation: Lowercase Greek indices such as α,β,γ, etc., will run from 1 to $m+n$, whereas lowercase Latin indices such as i,j,k,s,t, etc., will run from 1 to m, lowercase Latin indices with a prime, such as i',j',k', etc., will run from $m+1$ to $m+n$. Quantities associated to (M_1,F_1) and (M_2,F_2) are denoted with upper indices 1 and 2, respectively, such as $\overset{1}{\Gamma}{}^i_{j;k}$, $\overset{2}{\Gamma}{}^{i'}_{j';k'}$.

The local coordinates (z,v) on $\widetilde{M}$ are transformed by the following rules

$$\tilde{z}^i=\tilde{z}^i(z^1,\cdots,z^m),\quad \tilde{z}^{i'}=\tilde{z}^{i'}(z^{m+1},\cdots,z^{m+n}),\quad \tilde{v}^i=\frac{\partial\tilde{z}^i}{\partial z^j}v^j,\quad \tilde{v}^{i'}=\frac{\partial\tilde{z}^{i'}}{\partial z^{j'}}v^{j'}.$$

For $\partial/\partial v^\alpha$, we have

$$\frac{\partial}{\partial v^i}=\frac{\partial \tilde{z}^j}{\partial z^i}\frac{\partial}{\partial \tilde{v}^j},\qquad \frac{\partial}{\partial v^{i'}}=\frac{\partial \tilde{z}^{j'}}{\partial z^{i'}}\frac{\partial}{\partial \tilde{v}^{j'}}.$$

A local frame for $T^{1,0}\widetilde{M}$ is given by $(\partial_1,\cdots,\partial_{m+n},\dot{\partial}_1,\cdots,\dot{\partial}_{m+n})$, where

$$\partial_\mu=\frac{\partial}{\partial z^\mu},\quad \mu=1,\cdots,m,m+1,\cdots,m+n,$$
$$\dot{\partial}_\mu=\frac{\partial}{\partial v^\mu},\quad \mu=1,\cdots,m,m+1,\cdots,m+n.$$

Denote $g=F_1^2$, $h=F_2^2$, so that $G=F^2=\lambda_1^2 g+\lambda_2^2 h$ and

$$g_{i\bar{j}}=\frac{\partial^2 g}{\partial v^i\partial\overline{v^j}},\qquad h_{i'\bar{j}'}=\frac{\partial^2 h}{\partial v^{i'}\partial\overline{v^{j'}}}.$$

The fundamental tensor matrix of F is given by

$$(G_{\alpha\bar{\beta}})=\Big(\frac{\partial^2 G}{\partial v^\alpha\partial\overline{v^\beta}}\Big)=\begin{pmatrix}\lambda_1^2 g_{i\bar{j}} & 0\\ 0 & \lambda_2^2 h_{i'\bar{j}'}\end{pmatrix}, \tag{2.4}$$

with its inverse matrix $(G^{\bar{\beta}\alpha})$ given by

$$(G^{\bar{\beta}\alpha})=\begin{pmatrix}\lambda_1^{-2} g^{\bar{j}i} & 0\\ 0 & \lambda_2^{-2} h^{\bar{j}'i'}\end{pmatrix}. \tag{2.5}$$

Let $\mathcal{V}^{1,0}$ be the holomorphic vertical vector subbundle of $T^{1,0}\widetilde{M}$, which is locally spanned by the nature frame fields $\{\frac{\partial}{\partial v^i},\frac{\partial}{\partial v^{i'}}\}$. Then, the complementary vector subbundle $\mathcal{H}^{1,0}$ to $\mathcal{V}^{1,0}$ in $T^{1,0}\widetilde{M}$ is locally spanned by $\{\delta_i,\delta_{i'}\}$, where

$$\delta_i=\tfrac{\partial}{\partial z^i}-\Gamma^j_{;i}\tfrac{\partial}{\partial v^j}-\Gamma^{j'}_{;i}\tfrac{\partial}{\partial v^{j'}},\qquad i=1,\cdots,m,$$
$$\delta_{i'}=\tfrac{\partial}{\partial z^{i'}}-\Gamma^j_{;i'}\tfrac{\partial}{\partial v^j}-\Gamma^{j'}_{;i'}\tfrac{\partial}{\partial v^{j'}},\qquad i'=m+1,\cdots,m+n.$$

$\mathcal{V}^{1,0}$ and $\mathcal{H}^{1,0}$ are called the doubly twisted vertical distribution and horizontal distribution on $T^{1,0}\widetilde{M}$, respectively. Thus the holomorphic tangent bundle $T^{1,0}\widetilde{M}$ admits the decomposition

$$T^{1,0}\widetilde{M}=\mathcal{H}^{1,0}\bigoplus\mathcal{V}^{1,0}. \tag{2.6}$$

3 Connections of DTP-complex Finsler manifold

In complex Finsler geometry, we always use the complex Chern-Finsler connection, but sometimes other connections such that complex Rund connection, complex Berwald connection, complex Hashiguchi connection and Rund type complex Finsler connection are more convenient. In this section, we briefly introduce these connections mentioned above, and then we derive these connections of DTP-complex Finsler manifold, which are expressed in terms of connections of its components.

The Chern-Finsler connection $D:\mathcal{X}(\mathcal{V}^{1,0})\to\mathcal{X}(T^*_{\mathbb{C}}\widetilde{M}\otimes\mathcal{V}^{1,0})$ associated to a strongly pseudoconvex complex Finsler metric F was first constructed in [17] and systemically studied in [1].

The Chern-Finsler complex nonlinear connection $\Gamma^{\alpha}_{;\mu}$ associate to a given strongly pseudoconvex complex Finsler metric F is characterized by

$$\Gamma^{\alpha}_{;\mu}=:G^{\bar{\nu}\alpha}G_{\bar{\nu};\mu}. \tag{3.1}$$

The connection 1-forms ω^{α}_{β} of D are given by

$$\omega^{\alpha}_{\beta}=\Gamma^{\alpha}_{\beta;\mu}dz^{\mu}+\Gamma^{\alpha}_{\beta\mu}\psi^{\mu}, \tag{3.2}$$

where

$$\Gamma^{\alpha}_{\beta;\mu}=G^{\bar{\nu}\alpha}\delta_{\mu}(G_{\beta\bar{\nu}}), \tag{3.3}$$

$$\Gamma^{\alpha}_{\beta\mu}=G^{\bar{\nu}\alpha}\dot{\partial}_{\mu}(G_{\beta\bar{\nu}}), \tag{3.4}$$

$$\delta_{\mu}=\partial_{\mu}-\Gamma^{\alpha}_{;\mu}\dot{\partial}_{\alpha},\qquad \psi^{\mu}=dv^{\mu}+\Gamma^{\mu}_{;\alpha}dz^{\alpha}. \tag{3.5}$$

Note that

$$\Gamma^{\alpha}_{\beta;\mu}=\dot{\partial}_{\beta}(\Gamma^{\alpha}_{;\mu}), \tag{3.6}$$

$\Gamma^{\alpha}_{\beta;\mu}$ and$\Gamma^{\alpha}_{\beta\mu}$ are called the horizontal and vertical coefficients of the Chern-Finsler connection associated to F, respectively.

Lemma 3.1. *Let $(M_1\times_{(\lambda_1,\lambda_2)}M_2,F)$ be a DTP-complex Finsler manifold of (M_1,F_1) and (M_2,F_2). Then the Chern-Finsler complex nonlinear connection coefficients associated to F are given by*

$$(\Gamma^{\alpha}_{;\mu})=\begin{pmatrix}\Gamma^{i}_{;k} & \Gamma^{i'}_{;k}\\ \Gamma^{i}_{;k'} & \Gamma^{i'}_{;k'}\end{pmatrix},$$

where

$$\Gamma^{i}_{;k}=\overset{1}{\Gamma}{}^{i}_{;k}+2\lambda_1^{-1}\frac{\partial\lambda_1}{\partial z^k}v^i,\qquad \Gamma^{i'}_{;k}=2\lambda_2^{-1}\frac{\partial\lambda_2}{\partial z^k}v^{i'},$$

$$\Gamma^{i}_{;k'}=2\lambda_1^{-1}\frac{\partial\lambda_1}{\partial z^{k'}}v^i,\qquad \Gamma^{i'}_{;k'}=\overset{2}{\Gamma}{}^{i'}_{;k'}+2\lambda_2^{-1}\frac{\partial\lambda_2}{\partial z^{k'}}v^{i'}.$$

Proof. By using (2.5) and (3.1), we have

$$\Gamma^{i}_{;k}=G^{\bar{j}i}G_{\bar{j};k}+G^{\bar{j}'i}G_{\bar{j}';k}=\lambda_1^{-2}g^{\bar{j}i}\lambda_1^2g_{\bar{j};k}+2\lambda_1^{-1}\frac{\partial\lambda_1}{\partial z^k}g^{\bar{j}i}g_{\bar{j}}=\overset{1}{\Gamma}{}^{i}_{;k}+2\lambda_1^{-1}\frac{\partial\lambda_1}{\partial z^k}v^i.$$

Similarly, we can obtain other equations of Lemma 3.1. □

Using (3.4), (3.6) and Lemma 3.1, after a straightforward computation, we have

Proposition 3.1. *Let $(M_1\times_{(\lambda_1,\lambda_2)}M_2,F)$ be a DTP-complex Finsler manifold of (M_1,F_1) and (M_2,F_2), the horizontal and vertical coefficients of the Chern-Finsler connection associated to F are given by*

$$\begin{aligned}
&\Gamma^i_{j;k}=\overset{1}{\Gamma}{}^i_{j;k}+2\lambda_1^{-1}\frac{\partial\lambda_1}{\partial z^k}\delta^i_j,\quad \Gamma^i_{j;k'}=2\lambda_1^{-1}\frac{\partial\lambda_1}{\partial z^{k'}}\delta^i_j,\\
&\Gamma^{i'}_{j';k'}=\overset{2}{\Gamma}{}^{i'}_{j';k'}+2\lambda_2^{-1}\frac{\partial\lambda_2}{\partial z^{k'}}\delta^{i'}_{j'},\quad \Gamma^{i'}_{j';k}=2\lambda_2^{-1}\frac{\partial\lambda_2}{\partial z^{k}}\delta^{i'}_{j'},\\
&\Gamma^i_{j';k}=\Gamma^i_{j';k'}=\Gamma^{i'}_{j;k}=\Gamma^{i'}_{j;k'}=0,\\
&\Gamma^i_{jk}=\overset{1}{\Gamma}{}^i_{jk},\qquad \Gamma^{i'}_{j'k'}=\overset{2}{\Gamma}{}^{i'}_{j'k'},\qquad \Gamma^i_{j'k}=\Gamma^i_{jk'}=\Gamma^i_{j'k'}=\Gamma^{i'}_{jk}=\Gamma^{i'}_{j'k}=\Gamma^{i'}_{jk'}=0.
\end{aligned}$$

The complex Rund connection associated to a strongly pseudoconvex complex Finsler metric F was first introduced in [27] and were systemically studied in [6] and [5]. Let $\hat{D}:\mathcal{X}(\mathcal{V}^{1,0})\to\mathcal{X}(T^*_{\mathbb{C}}\tilde{M}\otimes\mathcal{V}^{1,0})$ be the complex Rund connection, then the connection 1-forms $\hat{\omega}$ of $\hat{D}$ are given by

$$\hat{\omega}^\alpha_\beta=\Gamma^\alpha_{\beta;\mu}dz^\mu, \tag{3.7}$$

where $\Gamma^\alpha_{\beta;\mu}$ are defined by (3.3). It is clearly from (3.2), $\hat{\omega}^\alpha_\beta$ are just the horizontal part of ω^α_β.

Using (3.3) and Lemma 3.1, we obtain

Proposition 3.2. *Let $(M_1\times_{(\lambda_1,\lambda_2)}M_2,F)$ be a DTP-complex Finsler manifold of (M_1,F_1) and (M_2,F_2), $\Gamma^\alpha_{\beta;\mu}$ be the coefficients of the complex Rund connection associated to F. Then*

$$\begin{aligned}
&\Gamma^i_{j;k}=\overset{1}{\Gamma}{}^i_{j;k}+2\lambda_1^{-1}\frac{\partial\lambda_1}{\partial z^k}\delta^i_j,\qquad \Gamma^i_{j;k'}=2\lambda_1^{-1}\frac{\partial\lambda_1}{\partial z^{k'}}\delta^i_j,\\
&\Gamma^{i'}_{j';k}=2\lambda_2^{-1}\frac{\partial\lambda_2}{\partial z^{k}}\delta^{i'}_{j'},\qquad \Gamma^{i'}_{j';k'}=\overset{2}{\Gamma}{}^{i'}_{j';k'}+2\lambda_2^{-1}\frac{\partial\lambda_2}{\partial z^{k'}}\delta^{i'}_{j'},\\
&\Gamma^i_{j';k}=\Gamma^i_{j';k'}=\Gamma^{i'}_{j;k}=\Gamma^{i'}_{j;k'}=0.
\end{aligned}$$

Next we shall consider the complex Berwald nonlinear connection. First we introduce the complex non-linear connection coefficients $\mathbb{G}^\alpha_\mu$, which is obtained from $\Gamma^\alpha_{;\mu}$. According to [22], the complex nonlinear connection coefficients $\Gamma^\alpha_{;\mu}$ always determine a complex spray $\mathbb{G}^\alpha=:\frac{1}{2}\Gamma^\alpha_{;\mu}v^\mu$. Conversely, the complex spray $\mathbb{G}^\alpha$ induces another complex nonlinear connection denoted by

$$\mathbb{G}^\alpha_\mu=\dot{\partial}_\mu(\mathbb{G}^\alpha), \tag{3.8}$$

which are called the complex Berwald nonlinear connection coefficients associated to F. Note that we get

$$\Gamma^\alpha_{;\mu}v^\mu=\mathbb{G}^\alpha_\mu v^\mu=2\mathbb{G}^\alpha.$$

The complex Berwald connection was first introduced in [22], and studied in [29,37]. Let $\breve{D}:\mathcal{X}(\mathcal{V}^{1,0})\to\mathcal{X}(T^*_{\mathbb{C}}\tilde{M}\otimes\mathcal{V}^{1,0})$ be a complex Berwald connection associated to strongly pseudoconvex complex Finsler metric F, its connection 1-form can be expressed as

$$\breve{\omega}^{\alpha}_{\beta}=\mathbb{G}^{\alpha}_{\beta\mu}dz^{\mu}, \tag{3.9}$$

where

$$\mathbb{G}^{\alpha}_{\beta\mu}=\dot{\partial}_{\beta}(\mathbb{G}^{\alpha}_{\mu}). \tag{3.10}$$

By Lemma 3.1, we can get

Lemma 3.2. *Let $(M_1\times_{(\lambda_1,\lambda_2)}M_2,F)$ be a DTP-complex Finsler manifold of (M_1,F_1) and (M_2,F_2). Then the complex spray coefficients $\mathbb{G}^{\alpha}$ associated to $\Gamma^{\alpha}_{;\mu}$ (or equivalently $\mathbb{G}^{\alpha}_{\mu}$) are given by*

$$\mathbb{G}^{i}=\overset{1}{\mathbb{G}}{}^{i}+\lambda_1^{-1}\Big(\frac{\partial\lambda_1}{\partial z^{s}}v^{s}+\frac{\partial\lambda_1}{\partial z^{s'}}v^{s'}\Big)v^{i}, \tag{3.11}$$

$$\mathbb{G}^{i'}=\overset{2}{\mathbb{G}}{}^{i'}+\lambda_2^{-1}\Big(\frac{\partial\lambda_2}{\partial z^{s}}v^{s}+\frac{\partial\lambda_2}{\partial z^{s'}}v^{s'}\Big)v^{i'}. \tag{3.12}$$

Using (3.8) and Lemma 3.2, by a straightforward computation, we have

Lemma 3.3. *Let $(M_1\times_{(\lambda_1,\lambda_2)}M_2,F)$ be a DTP-complex Finsler manifold of (M_1,F_1) and (M_2,F_2). Then the complex Berwald nonlinear connection coefficients $\mathbb{G}^{\alpha}_{\mu}$ are given by*

$$(\mathbb{G}^{\alpha}_{\mu})=\begin{pmatrix}\mathbb{G}^{i}_{k} & \mathbb{G}^{i'}_{k}\\ \mathbb{G}^{i}_{k'} & \mathbb{G}^{i'}_{k'}\end{pmatrix},$$

where

$$\mathbb{G}^{i}_{k}=\overset{1}{\mathbb{G}}{}^{i}_{k}+\lambda_1^{-1}\Big[\Big(\frac{\partial\lambda_1}{\partial z^{s}}v^{s}+\frac{\partial\lambda_1}{\partial z^{s'}}v^{s'}\Big)\delta^{i}_{k}+\frac{\partial\lambda_1}{\partial z^{k}}v^{i}\Big],\qquad \mathbb{G}^{i}_{k'}=\lambda_1^{-1}\frac{\partial\lambda_1}{\partial z^{k'}}v^{i},$$
$$\mathbb{G}^{i'}_{k'}=\overset{2}{\mathbb{G}}{}^{i'}_{k'}+\lambda_2^{-1}\Big[\Big(\frac{\partial\lambda_2}{\partial z^{s}}v^{s}+\frac{\partial\lambda_2}{\partial z^{s'}}v^{s'}\Big)\delta^{i'}_{k'}+\frac{\partial\lambda_2}{\partial z^{k'}}v^{i'}\Big],\quad \mathbb{G}^{i'}_{k}=\lambda_2^{-1}\frac{\partial\lambda_2}{\partial z^{k}}v^{i'}.$$

Using (3.10) and Lemma 3.3, by a straightforward computation, we obtain

Proposition 3.3. *Let $(M_1\times_{(\lambda_1,\lambda_2)}M_2,F)$ be a DTP-complex Finsler manifold of (M_1,F_1) and (M_2,F_2). Then the complex Berwald connection coefficients $\mathbb{G}^{\alpha}_{\beta\mu}$ associated to F are given by*

$$\mathbb{G}^{i}_{jk}=\overset{1}{\mathbb{G}}{}^{i}_{jk}+\lambda_1^{-1}\Big(\frac{\partial\lambda_1}{\partial z^{k}}\delta^{i}_{j}+\frac{\partial\lambda_1}{\partial z^{j}}\delta^{i}_{k}\Big),\qquad \mathbb{G}^{i}_{j'k}=\mathbb{G}^{i}_{kj'}=\lambda_1^{-1}\frac{\partial\lambda_1}{\partial z^{j'}}\delta^{i}_{k},\quad \mathbb{G}^{i}_{j'k'}=0,$$
$$\mathbb{G}^{i'}_{j'k'}=\overset{2}{\mathbb{G}}{}^{i'}_{j'k'}+\lambda_2^{-1}\Big(\frac{\partial\lambda_2}{\partial z^{k'}}\delta^{i'}_{j'}+\frac{\partial\lambda_2}{\partial z^{j'}}\delta^{i'}_{k'}\Big),\quad \mathbb{G}^{i'}_{j'k}=\mathbb{G}^{i'}_{kj'}=\lambda_2^{-1}\frac{\partial\lambda_2}{\partial z^{k}}\delta^{i'}_{j'},\quad \mathbb{G}^{i'}_{jk}=0.$$

The complex Hashiguchi connection $\check{D}:\mathcal{X}(\mathcal{V}^{1,0})\to\mathcal{X}(T^*_{\mathbb{C}}\tilde{M}\otimes\mathcal{V}^{1,0})$ is a complex analogue of the Hashiguchi connection in real Finsler geometry [20]. Recently Sun and Zhong gave a characterization of complex Hashiguchi connection in [29]. Its connection 1-forms $\check{\omega}^\alpha_\beta$ are given by

$$\check{\omega}^\alpha_\beta=\mathbb{G}^\alpha_{\beta\mu}dz^\mu+\Gamma^\alpha_{\beta\mu}\check{\psi}^\mu, \tag{3.13}$$

where $\mathbb{G}^\alpha_{\beta\mu}$ are given by (3.10), $\Gamma^\alpha_{\beta\mu}$ are given by (3.4) and $\check{\psi}^\mu=dv^\mu+\mathbb{G}^\mu_\alpha dz^\alpha$.

Proposition 3.4. *Let* $(M_1\times_{(\lambda_1,\lambda_2)}M_2,F)$ *be a DTP-complex Finsler manifold of* (M_1,F_1) *and* (M_2,F_2)*. Then the horizontal and vertical connection coefficients of the complex Hashiguchi connection, respectively* $\Gamma^\alpha_{\beta\mu}$ *and* $\mathbb{G}^\alpha_{\beta\mu}$ *are given by Proposition 3.1 and Proposition 3.3.*

In [2], Aldea and Munteanu gave the definition of complex Landsberg space. A complex Finsler manifold (M,F) is called a complex Landsberg manifold if

$$\mathbb{G}^\gamma_{\nu\mu}=\mathbb{L}^\gamma_{\nu\mu}, \tag{3.14}$$

where $\mathbb{L}^\gamma_{\nu\mu}$ are the horizontal connection coefficients of the Rund type complex linear connection [22], i.e.,

$$\mathbb{L}^\gamma_{\nu\mu}=\frac{1}{2}G^{\overline{\tau}\gamma}(\check{\delta}_\nu G_{\mu\overline{\tau}}+\check{\delta}_\mu G_{\nu\overline{\tau}}), \tag{3.15}$$

where

$$\check{\delta}_\nu=\partial_\nu-\mathbb{G}^\alpha_\nu\dot{\partial}_\alpha. \tag{3.16}$$

Proposition 3.5. *Let* $(M_1\times_{(\lambda_1,\lambda_2)}M_2,F)$ *be a DTP-complex Finsler manifold of* (M_1,F_1) *and* (M_2,F_2)*. Then*

$$\begin{aligned}
&\mathbb{L}^i_{jk}=\overset{1}{\mathbb{L}}{}^i_{jk}-\lambda_1^{-1}\Big(\frac{\partial\lambda_1}{\partial z^s}v^s+\frac{\partial\lambda_1}{\partial z^{s'}}v^{s'}\Big)\overset{1}{\Gamma}{}^i_{jk}, \qquad \mathbb{L}^i_{j'k}=\mathbb{L}^i_{kj'}=\lambda_1^{-1}\frac{\partial\lambda_1}{\partial z^{j'}}\delta^i_k, \quad \mathbb{L}^i_{j'k'}=0,\\
&\mathbb{L}^{i'}_{j'k'}=\overset{2}{\mathbb{L}}{}^{i'}_{j'k'}-\lambda_2^{-1}\Big(\frac{\partial\lambda_2}{\partial z^s}v^s+\frac{\partial\lambda_2}{\partial z^{s'}}v^{s'}\Big)\overset{2}{\Gamma}{}^{i'}_{j'k'}, \quad \mathbb{L}^{i'}_{j'k}=\mathbb{L}^{i'}_{kj'}=\lambda_2^{-1}\frac{\partial\lambda_2}{\partial z^k}\delta^{i'}_{j'}, \quad \mathbb{L}^{i'}_{jk}=0.
\end{aligned}$$

Proof. Using (2.4), (2.5), (3.15), (3.16) and Lemma 3.3, we have

$$\begin{aligned}
\mathbb{L}^i_{jk}&=\frac{1}{2}G^{\bar{l}i}(\check{\delta}_kG_{j\bar{l}}+\check{\delta}_jG_{k\bar{l}})+\frac{1}{2}G^{\bar{l}'i}(\check{\delta}_kG_{j\bar{l}'}+\check{\delta}_jG_{k\bar{l}'})\\
&=\frac{1}{2}G^{\bar{l}i}[(\partial_k-\mathbb{G}^m_k\dot{\partial}_m-\mathbb{G}^{m'}_k\dot{\partial}_{m'})G_{j\bar{l}}+(\partial_j-\mathbb{G}^n_j\dot{\partial}_n-\mathbb{G}^{n'}_j\dot{\partial}_{n'})G_{k\bar{l}}]\\
&=\frac{1}{2}G^{\bar{l}i}[(\partial_k-\overset{1}{\mathbb{G}}{}^m_k\dot{\partial}_m)G_{j\bar{l}}+(\partial_j-\overset{1}{\mathbb{G}}{}^n_j\dot{\partial}_n)G_{k\bar{l}}]-\lambda_1^{-1}\Big(\frac{\partial\lambda_1}{\partial z^s}v^s+\frac{\partial\lambda_1}{\partial z^{s'}}v^{s'}\Big)G^{\bar{l}i}G_{j\bar{l}k}\\
&=\frac{1}{2}\lambda_1^{-2}g^{\bar{l}i}\lambda_1^2(\overset{1}{\check{\delta}}_kg_{j\bar{l}}+\overset{1}{\check{\delta}}_jg_{k\bar{l}})-\lambda_1^{-1}\Big(\frac{\partial\lambda_1}{\partial z^s}v^s+\frac{\partial\lambda_1}{\partial z^{s'}}v^{s'}\Big)\lambda_1^{-2}g^{\bar{l}i}\lambda_1^2g_{j\bar{l}k}\\
&=\overset{1}{\mathbb{L}}{}^i_{jk}-\lambda_1^{-1}\Big(\frac{\partial\lambda_1}{\partial z^s}v^s+\frac{\partial\lambda_1}{\partial z^{s'}}v^{s'}\Big)\overset{1}{\Gamma}{}^i_{jk}.
\end{aligned}$$

Where in the third equality we use $g_{j\bar{l}m}v^m=0$, in the last equality we use $g^{\bar{l}i}g_{j\bar{l}k}=\overset{1}{\Gamma}{}^i_{jk}$.

Similarly, we can obtain other equations of Proposition 3.5. □

4 Holomorphic curvature and Ricci scalar curvature of DTP-complex Finsler manifolds

One of the fundamental problems in Finsler geometry is to describe the Finsler metric with constant holomorphic curvature and Ricci scalar curvature. Our objective in this section is to derive formulae of the holomorphic curvature and Ricci scalar curvature of DTP-complex Finsler manifold, which are expressed by the holomorphic curvature and Ricci scalar curvature of its components.

The definition of the holomorphic curvature on (M,F) from the curvature tensor Ω of a complex Finsler connection was considered by Kobayashi [17], and locally expression of the holomorphic curvature of a strongly pseudoconvex Finsler metric F along $v\in T^{1,0}_{z_0}M$ with respect to Chern-Finsler connection D is given by Abate and Patrizio [1]:

$$K_F(v)=-\frac{2}{G^2}G_\alpha\delta_{\bar{v}}(\Gamma^\alpha_{;\mu})v^\mu\overline{v^\nu}. \tag{4.1}$$

In [29], Sun and Zhong proved that the holomorphic curvature and Ricci scalar curvature of a strongly pseudoconvex Finsler metric F along a nonzero vector $v\in T^{1,0}_{z_0}M$ with respect to the Chern-Finsler connection D, or the complex Rund connection $\hat{D}$, or the complex Berwald connection $\breve{D}$, or the complex Hashiguchi connection $\check{D}$ are coincide with each other. Therefore, we use the Chern-Finsler connection to derive the holomorphic curvature of the DTP-complex Finsler manifold in this section.

Theorem 4.1. *Let $(M_1\times_{(\lambda_1,\lambda_2)}M_2,F)$ be a DTP-complex Finsler manifold of (M_1,F_1) and (M_2,F_2). Then the holomorphic curvature of (M,F) along a holomorphic tangent vector $v=(v^i,v^{i'})\in T^{1,0}_zM$ satisfying $F_1(\pi_1(v))=1$ and $F_2(\pi_2(v))=1$ is given by*

$$K_F(v)=\frac{\lambda_1^2g^2}{G^2}K_{F_1}(v_1)-\frac{4}{G^2}\left[\lambda_1^2g\partial_{\bar{v}}(\partial_\mu(ln\lambda_1)+\lambda_2^2h\partial_{\bar{v}}(\partial_\mu(ln\lambda_2)\right]v^\mu\overline{v^\nu}+\frac{\lambda_2^2h^2}{G^2}K_{F_2}(v_2). \tag{4.2}$$

Proof. According to (4.1), the holomorphic curvature of the DTP-complex Finsler manifold $(M_1\times_{(\lambda_1,\lambda_2)}M_2,F)$ with respect to Chern-Finsler connection is given by

$$K_F(v)=-\frac{2}{G^2}\left[G_i\delta_{\bar{v}}(\Gamma^i_{;j})v^j\overline{v^\nu}+G_i\delta_{\bar{v}}(\Gamma^i_{;j'})v^{j'}\overline{v^\nu}+G_{i'}\delta_{\bar{v}}(\Gamma^{i'}_{;j})v^j\overline{v^\nu}+G_{i'}\delta_{\bar{v}}(\Gamma^{i'}_{;j'})v^{j'}\overline{v^\nu}\right]. \tag{4.3}$$

Since $\overset{1}{\Gamma}{}^i_{;j}$ depend only on z^i, v^i and are holomorphic with respect to $v^{k'}$, we have

$$\begin{aligned}
-\frac{2}{G^2}G_i\delta_{\bar{v}}(\Gamma^i_{;j})v^j\overline{v^\nu} =& -\frac{2}{G^2}\lambda_1^2 g_i\delta_{\bar{k}}(\overset{1}{\Gamma}{}^i_{;j})v^j\overline{v^k} - \frac{2}{G^2}\lambda_1^2 g_i\delta_{\bar{k}}(2\lambda_1^{-1}\frac{\partial\lambda_1}{\partial z^j}v^i)v^j\overline{v^k}\\
&-\frac{2}{G^2}\lambda_1^2 g_i\delta_{\bar{k}'}(\overset{1}{\Gamma}{}^i_{;j})v^j\overline{v^{k'}} - \frac{2}{G^2}\lambda_1^2 g_i\delta_{\bar{k}'}(2\lambda_1^{-1}\frac{\partial\lambda_1}{\partial z^j}v^i)v^j\overline{v^{k'}}\\
=& \frac{\lambda_1^2 g^2}{G^2}K_{F_1}(v_1) - \frac{4}{G^2}\lambda_1^2 g(\frac{\partial^2 ln\lambda_1}{\partial z^j\partial\overline{z^k}}v^j\overline{v^k} + \frac{\partial^2 ln\lambda_1}{\partial z^j\partial\overline{z^{k'}}}v^j\overline{v^{k'}}),
\end{aligned} \tag{4.4}$$

where in the last equality we use

$$\begin{aligned}
\delta_{\bar{k}'}(\overset{1}{\Gamma}{}^i_{;j})v^j =& -\overline{\Gamma^l_{;k'}}\frac{\partial}{\partial\overline{v^l}}(\overset{1}{\Gamma}{}^i_{;j})v^j = 2\lambda_1^{-1}\frac{\partial\lambda_1}{\partial\overline{z^{k'}}}\overline{v^l}\frac{\partial}{\partial\overline{v^l}}(g^{\bar{s}i}g_{\bar{s};j})v^j\\
=& 2\lambda_1^{-1}\frac{\partial\lambda_1}{\partial\overline{z^{k'}}}\overline{v^l}\Big(-g^{\bar{t}i}g^{\bar{s}r}g_{r\bar{t}\bar{l}}h_{\bar{s};j} + g^{\bar{s}i}g_{\bar{s}\bar{l};j}\Big)v^j = 0.
\end{aligned}$$

Similarly, we obtain

$$-\frac{2}{G^2}G_i\delta_{\bar{v}}(\Gamma^i_{;j'})v^{j'}\overline{v^\nu} = -\frac{4}{G^2}\lambda_1^2 g(\frac{\partial^2 ln\lambda_1}{\partial z^{j'}\partial\overline{z^k}}v^{j'}\overline{v^k} + \frac{\partial^2 ln\lambda_1}{\partial z^{j'}\partial\overline{z^{k'}}}v^{j'}\overline{v^{k'}}), \tag{4.5}$$

$$-\frac{2}{G^2}G_{i'}\delta_{\bar{v}}(\Gamma^{i'}_{;j})v^j\overline{v^\nu} = -\frac{4}{G^2}\lambda_2^2 h(\frac{\partial^2 ln\lambda_2}{\partial z^j\partial\overline{z^k}}v^j\overline{v^k} + \frac{\partial^2 ln\lambda_2}{\partial z^j\partial\overline{z^{k'}}}v^j\overline{v^{k'}}), \tag{4.6}$$

$$-\frac{2}{G^2}G_{i'}\delta_{\bar{v}}(\Gamma^{i'}_{;j'})v^{j'}\overline{v^\nu} = \frac{\lambda_2^2 h^2}{G^2}K_{F_2}(v_2) - \frac{4}{G^2}\lambda_2^2 h(\frac{\partial^2 ln\lambda_2}{\partial z^{j'}\partial\overline{z^k}}v^{j'}\overline{v^k} + \frac{\partial^2 ln\lambda_2}{\partial z^{j'}\partial\overline{z^{k'}}}v^{j'}\overline{v^{k'}}). \tag{4.7}$$

Submit (4.4)-(4.7) into (4.3), we arrive at (4.2). □

Corollary 4.2. Let $(M_1\times_{(\lambda_1,\lambda_2)}M_2,F)$ be a DTP-complex Finsler manifold of (M_1,F_1) and (M_2,F_2). If $K_{F_1}(\pi_1(v)) = K_{F_2}(\pi_2(v)) = c$, $\ln\lambda_1$ and $\ln\lambda_2$ are pluriharmonic functions on $M_1\times M_2$. Then the holomorphic sectional curvature of $(M_1\times_{(\lambda_1,\lambda_2)}M_2,F)$ along $v=(v^i,v^{i'})$ is $K_F(v)\equiv c$.

The Ricci scalar curvature of F along a nonzero vector $v\in T^{1,0}_{z_0}(M_1\times M_2)$ associated to the Chern-Finsler connection is given by [22]:

$$Ric_F(v) = -\overline{\chi}(\Gamma^\mu_{;\mu}) = -\delta_{\overline{\alpha}}(\Gamma^\mu_{;\mu})v^{\overline{\alpha}}. \tag{4.8}$$

Theorem 4.3. *Let $(M_1\times_{(\lambda_1,\lambda_2)}M_2,F)$ be a DTP-complex Finsler manifold of (M_1,F_1) and (M_2,F_2). Then the Ricci scalar curvature of $(M_1\times_{(\lambda_1,\lambda_2)}M_2,F)$ associated to the Chern-Finsler connection is given by*

$$Ric_F = Ric_{F_1} + Ric_{F_2} - 2\partial_{\bar{\mu}}(\partial_k(ln\lambda_1))v^k\overline{v^\mu} - 2\partial_{\bar{\mu}}(\partial_{k'}(ln\lambda_2))v^{k'}\overline{v^\mu}, \tag{4.9}$$

where Ric_{F_1} and Ric_{F_2} are Ricci scalar curvatures of (M_1,F_1) and (M_2,F_2), respectively.

Proof. According to (4.8), the Ricci scalar curvature of the DTP-complex Finsler manifold $(M_1\times_{(\lambda_1,\lambda_2)}M_2,F)$ is given by

$$\mathrm{Ric}_F(v)= -\overline{v^i}\delta_{\bar{i}}(\Gamma^k_{;k})-\overline{v^{i'}}\delta_{\bar{i}'}(\Gamma^k_{;k})-\overline{v^i}\delta_{\bar{i}}(\Gamma^{k'}_{;k'})-\overline{v^{i'}}\delta_{\bar{i}'}(\Gamma^{k'}_{;k'}). \tag{4.10}$$

Since $\overset{1}{\Gamma}{}^k_{;k}$ depend only on z^i, v^i, using Lemma 3.1, we can get

$$\begin{aligned}-\overline{v^i}\delta_{\bar{i}}(\Gamma^k_{;k})&= -\overline{v^i}\Big(\frac{\partial}{\partial\overline{z^i}}-\overline{\Gamma^j_{;i}}\frac{\partial}{\partial\overline{v^j}}-\overline{\Gamma^{j'}_{;i}}\frac{\partial}{\partial\overline{v^{j'}}}\Big)(\overset{1}{\Gamma}{}^k_{;k}+2\lambda_1^{-1}\frac{\partial\lambda_1}{\partial z^k}v^k)\\&=\mathrm{Ric}_{F_1}-2\frac{\partial^2 ln\lambda_1}{\partial z^k\partial\overline{z^i}}v^k\overline{v^i}.\end{aligned} \tag{4.11}$$

Similarly, and note that

$$\overline{v^{i'}}\Big(-\overline{\Gamma^j_{;i'}}\frac{\partial}{\partial\overline{v^j}}\Big)(\Gamma^k_{;k})=0,\qquad \overline{v^i}\Big(-\overline{\Gamma^{j'}_{;i}}\frac{\partial}{\partial\overline{v^{j'}}}\Big)(\Gamma^{k'}_{;k'})=0,$$

we can obtain

$$-\overline{v^{i'}}\delta_{\bar{i}'}(\Gamma^k_{;k})=-2\frac{\partial^2 ln\lambda_1}{\partial z^k\partial\overline{z^{i'}}}v^k\overline{v^{i'}}, \tag{4.12}$$

$$-\overline{v^i}\delta_{\bar{i}}(\Gamma^{k'}_{;k'})=-2\frac{\partial^2 ln\lambda_2}{\partial z^{k'}\partial\overline{z^i}}v^{k'}\overline{v^i}, \tag{4.13}$$

$$-\overline{v^{i'}}\delta_{\bar{i}'}(\Gamma^{k'}_{;k'})=\mathrm{Ric}_{F_2}-2\frac{\partial^2 ln\lambda_2}{\partial z^{k'}\partial\overline{z^{i'}}}v^{k'}\overline{v^{i'}}. \tag{4.14}$$

Submit (4.11)-(4.14) into (4.10), we arrive at (4.9). □

5 Geodesics of DTP-complex Finsler manifold

In this section, we shall investigate real geodesics of DTP-complex Finsler manifold.

A real geodesic $\sigma^\alpha(t)$ with an affine parameter t of a complex Finsler manifold (M,F) satisfies the following equation [1]:

$$\ddot{\sigma}^\alpha(t)+\Gamma^\alpha_{;\mu}\dot{\sigma}^\mu=G^{\bar{\nu}\alpha}G_{\beta\bar{\gamma}}(\overline{\Gamma^\gamma_{\mu;\nu}}-\overline{\Gamma^\gamma_{\nu;\mu}})\dot{\sigma}^\beta\overline{\dot{\sigma}^\mu}. \tag{5.1}$$

Proposition 5.1. *Let $(M_1\times_{(\lambda_1,\lambda_2)}M_2,F)$ be a DTP-complex Finsler manifold of (M_1,F_1) and*

(M_2,F_2). *If* $\sigma(t)=(\sigma^\alpha(t))=(\sigma^k(t),\sigma^{k'}(t))$ *is a geodesic of* $(M_1\times_{(\lambda_1,\lambda_2)}M_2,F)$, *then* $\sigma(t)$ *satisfies*

$$\ddot\sigma^k(t)+\Gamma^k_{;\mu}\dot\sigma^\mu=g^{\bar sk}g_{i\bar l}(\overline{\overset{1}{\Gamma^l_{r;s}}}-\overline{\overset{1}{\Gamma^l_{s;r}}})\dot\sigma^i\overline{\dot\sigma^r}-2\lambda_1^{-1}\frac{\partial\lambda_1}{\partial\overline{z^{r'}}}\overline{\dot\sigma^{r'}}v^k$$
$$+\lambda_1^{-2}g^{\bar sk}\Big[2\lambda_1(g\frac{\partial\lambda_1}{\partial\overline{z^s}}-g_{\bar s}\frac{\partial\lambda_1}{\partial\overline{z^r}}\overline{\dot\sigma^r})+2\lambda_2h\frac{\partial\lambda_2}{\partial\overline{z^s}}\Big],\tag{5.2}$$

$$\ddot\sigma^{k'}(t)+\Gamma^{k'}_{;\mu}\dot\sigma^\mu=h^{\bar s'k'}h_{i'\bar l'}(\overline{\overset{2}{\Gamma^{l'}_{r';s'}}}-\overline{\overset{2}{\Gamma^{l'}_{s';r'}}})\dot\sigma^{i'}\overline{\dot\sigma^{r'}}-2\lambda_2^{-1}\frac{\partial\lambda_2}{\partial\overline{z^r}}\overline{\dot\sigma^r}v^{k'}$$
$$+\lambda_2^{-2}h^{\bar s'k'}\Big[2\lambda_2(h\frac{\partial\lambda_2}{\partial\overline{z^{s'}}}-h_{\bar s'}\frac{\partial\lambda_2}{\partial\overline{z^{r'}}}\overline{\dot\sigma^{r'}})+2\lambda_1g\frac{\partial\lambda_1}{\partial\overline{z^{s'}}}\Big].\tag{5.3}$$

Proof. Let $\alpha=k$ in (5.1), we have

$$\begin{aligned}\ddot\sigma^k(t)+\Gamma^k_{;\mu}\dot\sigma^\mu&=G^{\bar vk}G_{\beta\bar\gamma}(\overline{\Gamma^\gamma_{\mu;\nu}}-\overline{\Gamma^\gamma_{\nu;\mu}})\dot\sigma^\beta\overline{\dot\sigma^\mu}\\
&=\lambda_1^{-2}g^{\bar sk}\Big[G_{i\bar l}(\overline{\Gamma^l_{r;s}}-\overline{\Gamma^l_{s;r}})\dot\sigma^i\overline{\dot\sigma^r}+G_{i\bar l}(\overline{\Gamma^l_{r';s}}-\overline{\Gamma^l_{s;r'}})\dot\sigma^i\overline{\dot\sigma^{r'}}\\
&\quad+G_{i'\bar l}(\overline{\Gamma^l_{r;s}}-\overline{\Gamma^l_{s;r}})\dot\sigma^{i'}\overline{\dot\sigma^r}+G_{i'\bar l}(\overline{\Gamma^l_{r';s}}-\overline{\Gamma^l_{s;r'}})\dot\sigma^{i'}\overline{\dot\sigma^{r'}}\\
&\quad+G_{i\bar l'}(\overline{\Gamma^{l'}_{r;s}}-\overline{\Gamma^{l'}_{s;r}})\dot\sigma^i\overline{\dot\sigma^r}+G_{i\bar l'}(\overline{\Gamma^{l'}_{r';s}}-\overline{\Gamma^{l'}_{s;r'}})\dot\sigma^i\overline{\dot\sigma^{r'}}\\
&\quad+G_{i'\bar l'}(\overline{\Gamma^{l'}_{r;s}}-\overline{\Gamma^{l'}_{s;r}})\dot\sigma^{i'}\overline{\dot\sigma^r}+G_{i'\bar l'}(\overline{\Gamma^{l'}_{r';s}}-\overline{\Gamma^{l'}_{s;r'}})\dot\sigma^{i'}\overline{\dot\sigma^{r'}}\Big].\end{aligned}\tag{5.4}$$

Submit (2.4), (2.5) and the equations of Proposition 3.1 into (5.4), after a standard computation, we obtain (5.2).

Similary, we have (5.3). □

Corollary 5.1. *Let* $(M_1\times_{(1,\lambda_2)}M_2,F)$ *be a twisted product complex Finsler manifold of* (M_1,F_1) *and* (M_2,F_2). *If* $\sigma(t)=(\sigma^\alpha(t))=(\sigma^k(t),\sigma^{k'}(t))$ *is a geodesic of* $(M_1\times_{(1,\lambda_2)}M_2,F)$, *then* $\sigma(t)$ *satisfies*

$$\ddot\sigma^k(t)+\Gamma^k_{;\mu}\dot\sigma^\mu=g^{\bar sk}g_{i\bar l}(\overline{\overset{1}{\Gamma^l_{r;s}}}-\overline{\overset{1}{\Gamma^l_{s;r}}})\dot\sigma^i\overline{\dot\sigma^r}+2\lambda_2hg^{\bar sk}\frac{\partial\lambda_2}{\partial\overline{z^s}},\tag{5.5}$$

$$\ddot\sigma^{k'}(t)+\Gamma^{k'}_{;\mu}\dot\sigma^\mu=h^{\bar s'k'}h_{i'\bar l'}(\overline{\overset{2}{\Gamma^{l'}_{r';s'}}}-\overline{\overset{2}{\Gamma^{l'}_{s';r'}}})\dot\sigma^{i'}\overline{\dot\sigma^{r'}}-2\lambda_2^{-1}\frac{\partial\lambda_2}{\partial\overline{z^r}}\overline{\dot\sigma^r}v^{k'}$$
$$+2\lambda_2^{-1}h^{\bar s'k'}(h\frac{\partial\lambda_2}{\partial\overline{z^{s'}}}-h_{\bar s'}\frac{\partial\lambda_2}{\partial\overline{z^{r'}}}\overline{\dot\sigma^{r'}}).\tag{5.6}$$

Next we will consider the case of $\lambda_1\equiv1$.

Theorem 5.2. *Let* $(M_1\times_{(1,\lambda_2)}M_2,F)$ *be a twisted product complex Finsler manifold of* (M_1,F_1) *and* (M_2,F_2).

(i) *If the twisted function* λ_2 *depends only on the point of* M_2, *then any geodesic of* (M_1,F_1) *is a geodesic of* $(M_1\times_{(1,\lambda_2)}M_2,F)$, *that is to say* (M_1,F_1) *is a totally geodesic subspace of the twisted product complex Finsler space.*

(ii) *If the twisted function λ_2 depends only on the point of M_2, then the projection of any geodesic of the twisted product complex Finsler manifold $(M_1\times_{(1,\lambda_2)}M_2,F)$ onto M_1 is a geodesic of (M_1,F_1).*

Proof. Since the proof of (ii) is similar to (i), we only prove (i). If $(\sigma^k(t))$ is a geodesic of (M_1,F_1), then

$$\ddot{\sigma}^k(t)+\overset{1}{\Gamma}{}^k_{;i}\,\dot{\sigma}^i \quad = \quad g^{\bar{s}k}g_{i\bar{l}}(\overline{\overset{1}{\Gamma}{}^l_{r;s}}-\overline{\overset{1}{\Gamma}{}^l_{s;r}})\dot{\sigma}^i\overline{\dot{\sigma}^r}. \tag{5.7}$$

Consider the curve $(\sigma^k(t),\sigma^{k'})$, where $\sigma^{k'}$ is a constant, therefore $\dot{\sigma}^{k'}=0$. Thus the differential equation (5.5) reduces to

$$\ddot{\sigma}^k(t)+\overset{1}{\Gamma}{}^k_{;i}\,\dot{\sigma}^i \quad = \quad g^{\bar{s}k}g_{i\bar{l}}(\overline{\overset{1}{\Gamma}{}^l_{r;s}}-\overline{\overset{1}{\Gamma}{}^l_{s;r}})\dot{\sigma}^i\overline{\dot{\sigma}^r}+2\lambda_1^{-2}\lambda_2 h g^{\bar{s}k}\frac{\partial\lambda_2}{\partial\overline{z^s}}, \tag{5.8}$$

where we use $\Gamma^k_{;\mu}\dot{\sigma}^\mu=\overset{1}{\Gamma}{}^k_{;i}\,\dot{\sigma}^i+\Gamma^k_{;i'}\dot{\sigma}^{i'}=\overset{1}{\Gamma}{}^k_{;i}\,\dot{\sigma}^i$. Notice that the twisted function λ_2 depends only on the point of M_2, i.e. $\frac{\partial\lambda_2}{\partial\overline{z^s}}=0$, therefore, the differential equation (5.7) coincides with (5.8). So that the curve $(\sigma^k(t),\sigma^{k'})$ is a geodesic of $(M_1\times_{(1,\lambda_2)}M_2,F)$. □

6 Doubly twisted product of special complex Finsler manifolds

In this section, we study the necessary and sufficient conditions that the DTP-complex Finsler manifold $(M_1\times_{(\lambda_1,\lambda_2)}M_2,F)$ is a Kähler Finsler (or weakly Kähler Finsler, complex Berwald, weakly complex Berwald, complex Landsberg) manifold. In particular, we provide a possible way to construct weakly complex Berwald metric and give a characterization for a complex non-Berwald Landsberg metric.

Definition 6.1 ([1]). *Let F be a strongly pseudoconvex complex Finsler metric on a complex manifold M. F is called a strongly Kähler Finsler metric iff $\Gamma^\alpha_{\mu;\beta}-\Gamma^\alpha_{\beta;\mu}=0$; called a Kähler Finsler metric iff $(\Gamma^\alpha_{\mu;\beta}-\Gamma^\alpha_{\beta;\mu})v^\mu=0$; called a weakly Kähler Finsler metric iff*

$$G_\alpha(\Gamma^\alpha_{\beta;\mu}-\Gamma^\alpha_{\mu;\beta})v^\beta=0. \tag{6.1}$$

In [12], Chen and Shen proved that Kähler Finsler metric and strongly Kähler Finsler metric actually coincide. Therefore there exist only two Kählerian notions in complex Finsler, i.e., Kähler Finsler metric and weakly Kähler Finsler metric.

Next we want to see when the DTP-complex Finsler manifold is Kähler Finsler manifold or weakly Kähler Finsler manifold.

Theorem 6.1. *Let $(M_1\times_{(\lambda_1,\lambda_2)}M_2,F)$ be a DTP-complex Finsler manifold of (M_1,F_1) and (M_2,F_2). Then $(M_1\times_{(\lambda_1,\lambda_2)}M_2,F)$ is a Kähler Finsler manifold if and only if the functions λ_1 and λ_2 depend only on the point of M_1 and M_2, respectively, and the following equations hold:*

$$\begin{cases} \overset{1}{\Gamma}{}^i_{j;k}+2\lambda_1^{-1}\dfrac{\partial\lambda_1}{\partial z^k}\delta^i_j=\overset{1}{\Gamma}{}^i_{k;j}+2\lambda_1^{-1}\dfrac{\partial\lambda_1}{\partial z^j}\delta^i_k,\\ \overset{2}{\Gamma}{}^{i'}_{j';k'}+2\lambda_2^{-1}\dfrac{\partial\lambda_2}{\partial z^{k'}}\delta^{i'}_{j'}=\overset{2}{\Gamma}{}^{i'}_{k';j'}+2\lambda_2^{-1}\dfrac{\partial\lambda_2}{\partial z^{j'}}\delta^{i'}_{k'}. \end{cases} \tag{6.2}$$

Proof. Assume that $(M_1\times_{(\lambda_1,\lambda_2)}M_2,F)$ is a Kähler Finsler manifold, then $\Gamma^\alpha_{\beta;\mu}=\Gamma^\alpha_{\mu;\beta}$. By Proposition 3.1, this is equivalent to

$$\overset{1}{\Gamma}{}^i_{j;k}+2\lambda_1^{-1}\frac{\partial\lambda_1}{\partial z^k}\delta^i_j=\overset{1}{\Gamma}{}^i_{k;j}+2\lambda_1^{-1}\frac{\partial\lambda_1}{\partial z^j}\delta^i_k,\quad \overset{2}{\Gamma}{}^{i'}_{j';k'}+2\lambda_2^{-1}\frac{\partial\lambda_1}{\partial z^{k'}}\delta^{i'}_{j'}=\overset{2}{\Gamma}{}^{i'}_{k';j'}+2\lambda_2^{-1}\frac{\partial\lambda_1}{\partial z^{j'}}\delta^{i'}_{k'}, \tag{6.3}$$

$$2\lambda_1^{-1}\frac{\partial\lambda_1}{\partial z^{k'}}\delta^i_j=\Gamma^i_{j;k'}=\Gamma^i_{k';j}=0,\qquad 2\lambda_2^{-1}\frac{\partial\lambda_2}{\partial z^k}\delta^{i'}_{j'}=\Gamma^{i'}_{j';k}=\Gamma^{i'}_{k;j'}=0. \tag{6.4}$$

(6.4) implies that the functions λ_1 and λ_2 depend only on the point of M_1 and M_2, respectively. Thus ends the proof. □

Corollary 6.2. *Let $(M_1\times_{(\lambda_1,\lambda_2)}M_2,F)$ be a DTP-complex Finsler manifold of two Kähler Finsler manifolds (M_1,F_1) and (M_2,F_2). Then $(M_1\times_{(\lambda_1,\lambda_2)}M_2,F)$ is a Kähler Finsler manifold if and only if the functions λ_1 and λ_2 are positive constants.*

Proof. Since (M_1,F_1) and (M_2,F_2) are Kähler Finsler manifolds, we have

$$\overset{1}{\Gamma}{}^i_{j;k}=\overset{1}{\Gamma}{}^i_{k;j},\quad \overset{2}{\Gamma}{}^{i'}_{j';k'}=\overset{2}{\Gamma}{}^{i'}_{k';j'}. \tag{6.5}$$

Thus system (6.2) simplified as:

$$\begin{cases} \dfrac{\partial\lambda_1}{\partial z^k}\delta^i_j=\dfrac{\partial\lambda_1}{\partial z^j}\delta^i_k, & (6.6)\\ \dfrac{\partial\lambda_2}{\partial z^{k'}}\delta^{i'}_{j'}=\dfrac{\partial\lambda_2}{\partial z^{j'}}\delta^{i'}_{k'}. & (6.7) \end{cases}$$

(6.6) implies $\frac{\partial\lambda_1}{\partial z^k}=0$, (6.7) implies $\frac{\partial\lambda_2}{\partial z^{k'}}=0$. Thus, we conclude that the functions λ_1 and λ_2 are positive constants.

□

Theorem 6.3. *Let $(M_1\times_{(\lambda_1,\lambda_2)}M_2,F)$ be a DTP-complex Finsler manifold of (M_1,F_1) and (M_2,F_2). Then $(M_1\times_{(\lambda_1,\lambda_2)}M_2,F)$ is a weakly Kähler Finsler manifold if and only if the following equations hold:*

$$\begin{cases} \lambda_1^2 g_i(\overset{1}{\Gamma}{}^i_{j;k}-\overset{1}{\Gamma}{}^i_{k;j})v^j=2\lambda_1 g_k\dfrac{\partial\lambda_1}{\partial z^{j'}}v^{j'}-2\lambda_2 h\dfrac{\partial\lambda_2}{\partial z^k}-2g_i\lambda_1(\dfrac{\partial\lambda_1}{\partial z^k}\delta^i_j-\dfrac{\partial\lambda_1}{\partial z^j}\delta^i_k)v^j, & (6.8)\\ \lambda_2^2 h_{i'}(\overset{2}{\Gamma}{}^{i'}_{j';k'}-\overset{2}{\Gamma}{}^{i'}_{k';j'})v^{j'}=2\lambda_2 h_{k'}\dfrac{\partial\lambda_2}{\partial z^j}v^j-2\lambda_1 g\dfrac{\partial\lambda_1}{\partial z^{k'}}-2h_{i'}\lambda_2(\dfrac{\partial\lambda_2}{\partial z^{k'}}\delta^{i'}_{j'}-\dfrac{\partial\lambda_2}{\partial z^{j'}}\delta^{i'}_{k'})v^{j'}. & (6.9) \end{cases}$$

Proof. By putting $\mu = k$ in (6.1) and using Proposition 3.1, after a long but trivial computation, we obtain

$$\begin{aligned}
&G_\alpha(\Gamma^\alpha_{\beta;k} - \Gamma^\alpha_{k;\beta})v^\beta \\
&= G_i(\Gamma^i_{j;k} - \Gamma^i_{k;j})v^j + G_i(\Gamma^i_{j';k} - \Gamma^i_{k;j'})v^{j'} + G_{i'}(\Gamma^{i'}_{j;k} - \Gamma^{i'}_{k;j})v^j + G_{i'}(\Gamma^{i'}_{j';k} - \Gamma^{i'}_{k;j'})v^{j'} \\
&= \lambda_1^2 g_i(\overset{1}{\Gamma}{}^i_{j;k} - \overset{1}{\Gamma}{}^i_{k;j})v^j + 2g_i\lambda_1\Big(\frac{\partial\lambda_1}{\partial z^k}\delta^i_j - \frac{\partial\lambda_1}{\partial z^j}\delta^i_k\Big)v^j - 2\lambda_1 g_i\frac{\partial\lambda_1}{\partial z^{j'}}v^{j'}\delta^i_k + 2\lambda_2 h_{i'}\frac{\partial\lambda_2}{\partial z^k}v^{j'}\delta^{i'}_{j'} \\
&= 0.
\end{aligned}$$

After simplify, and notice that $h_{i'}v^{j'}\delta^{i'}_{j'} = h$, we can get (6.8).

Similarly, we can obtain (6.9). □

Corollary 6.4. *Let $(M_1 \times_{(1,\lambda_2)} M_2, F)$ be a twisted product complex Finsler manifold of (M_1,F_1) and (M_2,F_2), the function λ_2 depends only on the point of M_2. Then $(M_1 \times_{(1,\lambda_2)} M_2, F)$ is a weakly Kähler Finsler manifold if and only if (M_1,F_1) is a weakly Kähler Finsler manifold and the following equation hold:*

$$\lambda_2 h_{i'}(\overset{2}{\Gamma}{}^{i'}_{j';k'} - \overset{2}{\Gamma}{}^{i'}_{k';j'})v^{j'} = 2\Big(h_{k'}\frac{\partial\lambda_2}{\partial z^{j'}}v^{j'} - h\frac{\partial\lambda_2}{\partial z^{k'}}\Big). \tag{6.10}$$

Corollary 6.5. *Let $(M_1 \times_{(\lambda_1,\lambda_2)} M_2, F)$ be a DTP-complex Finsler manifold of two weakly Kähler Finsler manifolds (M_1,F_1) and (M_2,F_2). Then $(M_1 \times_{(\lambda_1,\lambda_2)} M_2, F)$ is a weakly Kähler Finsler manifold if and only if the functions λ_1 and λ_2 are positive constants.*

Proof. Since (M_1,F_1) and (M_2,F_2) are weakly Kähler Finsler manifolds, we have

$$g_i(\overset{1}{\Gamma}{}^i_{k;j} - \overset{1}{\Gamma}{}^i_{j;k})v^k = 0, \quad h_{i'}(\overset{2}{\Gamma}{}^{i'}_{j';k'} - \overset{2}{\Gamma}{}^{i'}_{k';j'})v^{j'} = 0.$$

Thus (6.8) and (6.9) simplified as:

$$\begin{cases}
\lambda_1 g_k\dfrac{\partial\lambda_1}{\partial z^{j'}}v^{j'} - \lambda_2 h\dfrac{\partial\lambda_2}{\partial z^k} - g_i\lambda_1\Big(\dfrac{\partial\lambda_1}{\partial z^k}\delta^i_j - \dfrac{\partial\lambda_1}{\partial z^j}\delta^i_k\Big)v^j = 0, & (6.11)\\
\lambda_2 h_{k'}\dfrac{\partial\lambda_2}{\partial z^j}v^j - \lambda_1 g\dfrac{\partial\lambda_1}{\partial z^{k'}} - h_{i'}\lambda_2\Big(\dfrac{\partial\lambda_2}{\partial z^{k'}}\delta^{i'}_{j'} - \dfrac{\partial\lambda_2}{\partial z^{j'}}\delta^{i'}_{k'}\Big)v^{j'} = 0. & (6.12)
\end{cases}$$

Note that g_k, λ_1 and λ_2 are independent of $v^{k'}$, thus differentiating (6.11) with respect to $v^{k'}$, we get

$$h_{k'}\lambda_2\frac{\partial\lambda_2}{\partial z^k} = g_k\lambda_1\frac{\partial\lambda_1}{\partial z^{k'}}. \tag{6.13}$$

Interchanging indices j and k in (6.11), and then contracting the obtained equality with v^j, we get

$$\lambda_1 g_j\frac{\partial\lambda_1}{\partial z^{k'}}v^{k'}v^j - \lambda_2 h\frac{\partial\lambda_2}{\partial z^j}v^j - 2g_i\lambda_1\Big(\frac{\partial\lambda_1}{\partial z^j}\delta^i_k - \frac{\partial\lambda_1}{\partial z^k}\delta^i_j\Big)v^k v^j = 0. \tag{6.14}$$

Contracting (6.12) with $v^{k'}$, we get

$$\lambda_2 h_{k'}\frac{\partial\lambda_2}{\partial z^j}v^j v^{k'} - \lambda_1 g\frac{\partial\lambda_1}{\partial z^{k'}}v^{k'} - h_{i'}\lambda_2\Big(\frac{\partial\lambda_2}{\partial z^{k'}}\delta^{i'}_{j'} - \frac{\partial\lambda_2}{\partial z^{j'}}\delta^{i'}_{k'}\Big)v^{k'}v^{j'} = 0. \tag{6.15}$$

Now subtracting (6.15) from (6.14) and using (6.13), we get

$$\lambda_2 h\frac{\partial\lambda_2}{\partial z^j}v^j = \lambda_1 g\frac{\partial\lambda_1}{\partial z^{k'}}v^{k'}. \tag{6.16}$$

Differentiating (6.16) with respect to $v^{k'}$, $\overline{v^{s'}}$ we get

$$h_{k'\bar{s}'}\lambda_2\frac{\partial\lambda_2}{\partial z^k} = 0.$$

Which implies $\frac{\partial\lambda_2}{\partial z^k} = 0$. Similarly, we get $\frac{\partial\lambda_1}{\partial z^{k'}} = 0$. Thus, (6.11) simplified as

$$g_i\Big(\frac{\partial\lambda_1}{\partial z^k}\delta^i_j - \frac{\partial\lambda_1}{\partial z^j}\delta^i_k\Big)v^j = 0. \tag{6.17}$$

Differentiating (6.17) with respect to $\overline{v^j}$, we have

$$g_{i\bar{j}}\Big(\frac{\partial\lambda_1}{\partial z^k}\delta^i_j - \frac{\partial\lambda_1}{\partial z^j}\delta^i_k\Big)v^j = 0. \tag{6.18}$$

Since $(g_{i\bar{j}})$ is a positive definite matrix, we have

$$\Big(\frac{\partial\lambda_1}{\partial z^k}\delta^i_j - \frac{\partial\lambda_1}{\partial z^j}\delta^i_k\Big)v^j = 0. \tag{6.19}$$

Differentiating (6.19) with respect to v^j, we get

$$\frac{\partial\lambda_1}{\partial z^k}\delta^i_j = \frac{\partial\lambda_1}{\partial z^j}\delta^i_k.$$

Which implies $\frac{\partial\lambda_1}{\partial z^k} = 0$. Similarly, we get $\frac{\partial\lambda_2}{\partial z^{k'}} = 0$. Thus, we conclude that the functions λ_1 and λ_2 are positive constants. □

As an immediate consequence of the relations (2.4), we have

Theorem 6.6. *The DTP-complex Finsler manifold* $(M_1 \times_{(\lambda_1,\lambda_2)} M_2, F)$ *is a Hermitian manifold if and only if* (M_1, F_1) *and* (M_2, F_2) *are Hermitian manifolds.*

Before starting to analyze the case of complex Berwald manifold, weakly complex Berwald manifold and complex Landsberg manifold, we need the following definitions.

Definition 6.2 ([5,7])**.** *A complex Finsler manifold* (M,F) *is said to be modeled on a complex Minkowski space if the connection coefficients* $\Gamma^\alpha_{\beta;\mu}(z,v)$ *of the Chern-Finsler connection depend only on the coordinates of the base manifold M, i.e.* $\Gamma^\alpha_{\beta;\mu} = \Gamma^\alpha_{\beta;\mu}(z)$.

Definition 6.3 ([7]). *Let (M,F) be a complex Finsler manifold, if the horizontal connection coefficients $\Gamma^{\alpha}_{\beta;\mu}(z,v)$ of the Chern-Finsler connection depend only on the coordinates of the base manifold M, i.e. $\Gamma^{\alpha}_{\beta;\mu}=\Gamma^{\alpha}_{\beta;\mu}(z)$, and its associated Hermitian metric h_F is a Kähler metric on M, then we call (M,F) a complex Berwald manifold.*

Remark 6.1. In [6], Aikou gave the definition of complex Berwald manifold in which without the Kähler condition. But different from real Finsler geometry, in complex Finsler geometry, there exist two different covariant derivatives for Cartan tensor, $C_{i\bar{j}k|h}$ and $C_{i\bar{j}k|\bar{h}}$ [5]. The requirement of the Kähler condition implies that $C_{i\bar{j}k|h}=0$. Later, Aikou gave the above definitions of complex Berwald manifold in [7], and widely used in various topics [2,5].

Definition 6.4 ([36]). *Let F be a strongly pseudoconvex complex Finsler metric on M. If locally the connection coefficients $\mathbb{G}^{\alpha}_{\beta\mu}(z,v)$ of the associated complex Berwald connection are independent of fibre coordinates v: $\mathbb{G}^{\alpha}_{\beta\mu}(z,v)=\mathbb{G}^{\alpha}_{\beta\mu}(z)$, then F is called a weakly complex Berwald metric.*

Definition 6.5 ([2]). *Let F be a complex Finsler metric on complex manifold M, F is said to be a complex Landsberg metric if it satisfies*

$$\mathbb{G}^{\gamma}_{\nu\mu}=\mathbb{L}^{\gamma}_{\nu\mu}. \tag{6.20}$$

According to Proposition 3.1, we have

Proposition 6.1. *The DTP-complex Finsler manifold $(M_1\times_{(\lambda_1,\lambda_2)}M_2,F)$ is modeled on a complex Minkowski space if and only if (M_1,F_1) and (M_2,F_2) are modeled on a complex Minkowski space.*

According to Definition 6.2 and Definition 6.3, Theorem 6.1 and Proposition 6.1, we obtain

Theorem 6.7. *The DTP-complex Finsler manifold $(M_1\times_{(\lambda_1,\lambda_2)}M_2,F)$ is a complex Berwald manifold if and only if (M_1,F_1) and (M_2,F_2) are modeled on complex Minkowski space and the system* (6.2) *hold.*

According to Proposition 3.1, Corolory 6.2 and Definition 6.3, we obtain

Corollary 6.8. *Let $(M_1\times_{(\lambda_1,\lambda_2)}M_2,F)$ be a DTP-complex Finsler manifold of two complex Berwald Finsler manifolds (M_1,F_1) and (M_2,F_2). Then $(M_1\times_{(\lambda_1,\lambda_2)}M_2,F)$ is a complex Berwald Finsler manifold if and only if the functions λ_1 and λ_2 are positive constants.*

If there exits an open cover $\{U,X_U\}$ such that on each $\pi_T^{-1}(U)$ the function F is a function of the fibre-coordinate only, then the complex Finsler manifold (M,F) will be a complex locally Minkowski [6]. The necessary and sufficient condition of the complex Finsler manifold (M,F) to be a complex locally Minkowski is that it is modeled on a complex Minkowski space and the complex Rund connection on (M,F) is holomorphic.

In short, the horizontal coefficients $\Gamma^{\alpha}_{\beta;\mu}(z,v)$ of the Chern-Finsler connection satisfies the following conditions:

$$\begin{cases} \Gamma^{\alpha}_{\beta;\mu}(z,v) &= \Gamma^{\alpha}_{\beta;\mu}(z), \\ \dfrac{\partial(\Gamma^{\alpha}_{\beta;\mu})}{\partial \overline{z^{\nu}}} &= 0. \end{cases}$$

In [7], Aikou established an example of complex manifold which is modeled on a complex Minkowski space, but not complex locally Minkowski.

According to Proposition 3.1 and Proposition 6.1, we have

Corollary 6.9. *If* $\ln\lambda_1$ *and* $\ln\lambda_2$ *are pluriharmonic functions on* $M_1\times M_2$*. Then the DTP-complex Finsler manifold* $(M_1\times_{(\lambda_1,\lambda_2)}M_2,F)$ *is a complex locally Minkowski if and only if* (M_1,F_1) *and* (M_2,F_2) *are complex locally Minkowski.*

According to the Definition 6.4 and Proposition 3.3, we obtain

Theorem 6.10. *The DTP-complex Finsler manifold* $(M_1\times_{(\lambda_1,\lambda_2)}M_2,F)$ *is a weakly complex Berwald manifold if and only if* (M_1,F_1) *and* (M_2,F_2) *are weakly complex Berwald manifolds.*

Remark 6.2. Note that a complex Berwald metric is actually a weakly complex Berwald metric, however, the converse is not true. For instance, under the assume that $\mathbb{G}^{\alpha}=\frac{1}{2}\Gamma^{\alpha}_{;\mu}v^{\mu}\equiv 0$ while $\Gamma^{\alpha}_{;\mu}$ do not vanish identically, the assertion that the complex Wrona metric is a weakly complex Berwald metric rather a complex Berwald metric was proved in [36]. It was also shown in [37] that there are lots of weakly complex Berwald metrics which are unitary invariant strongly pseudoconvex complex Finsler metric. Theorem 6.10 provide us an effective way to construct weakly complex Berwald manifolds.

According to definition 6.5, Proposition 3.3 and Proposition 3.5, we obtain

Theorem 6.11. *The DTP-complex Finsler manifold* $(M_1\times_{(\lambda_1,\lambda_2)}M_2,F)$ *is a complex Landsberg manifold if and only if the following equations hold:*

$$\begin{cases} \overset{1}{\mathbb{G}}{}^{i}_{jk}+\lambda_1^{-1}(\dfrac{\partial\lambda_1}{\partial z^k}\delta^i_j+\dfrac{\partial\lambda_1}{\partial z^j}\delta^i_k)=\overset{1}{\mathbb{L}}{}^{i}_{jk}-\lambda_1^{-1}\overset{1}{\Gamma}{}^{i}_{jk}(\dfrac{\partial\lambda_1}{\partial z^s}v^s+\dfrac{\partial\lambda_1}{\partial z^{s'}}v^{s'}), \\ \overset{2}{\mathbb{G}}{}^{i'}_{j'k'}+\lambda_2^{-1}(\dfrac{\partial\lambda_2}{\partial z^{k'}}\delta^{i'}_{j'}+\dfrac{\partial\lambda_2}{\partial z^{j'}}\delta^{i'}_{k'})=\overset{2}{\mathbb{L}}{}^{i'}_{j'k'}-\lambda_2^{-1}\overset{2}{\Gamma}{}^{i'}_{j'k'}(\dfrac{\partial\lambda_2}{\partial z^s}v^s+\dfrac{\partial\lambda_2}{\partial z^{s'}}v^{s'}). \end{cases} \tag{6.21}$$

Corollary 6.12. *Let* $(M_1\times_{(\lambda_1,\lambda_2)}M_2,F)$ *be a DTP-complex Finsler manifold of two Landsberg manifolds* (M_1,F_1) *and* (M_2,F_2)*. Then* $(M_1\times_{(\lambda_1,\lambda_2)}M_2,F)$ *is a Landsberg manifold if and only if the following equations hold:*

$$\begin{cases} \dfrac{\partial\lambda_1}{\partial z^k}\delta^i_j+\dfrac{\partial\lambda_1}{\partial z^j}\delta^i_k=-\overset{1}{\Gamma}{}^{i}_{jk}(\dfrac{\partial\lambda_1}{\partial z^s}v^s+\dfrac{\partial\lambda_1}{\partial z^{s'}}v^{s'}), \\ \dfrac{\partial\lambda_2}{\partial z^{k'}}\delta^{i'}_{j'}+\dfrac{\partial\lambda_2}{\partial z^{j'}}\delta^{i'}_{k'}=-\overset{2}{\Gamma}{}^{i'}_{j'k'}(\dfrac{\partial\lambda_2}{\partial z^s}v^s+\dfrac{\partial\lambda_2}{\partial z^{s'}}v^{s'}). \end{cases} \tag{6.22}$$

Remark 6.3. In real Finsler geometry, every Berwald space is a Landsberg space, the converse, however, has been a long-standing open problem [8]. This problem was studied by several authors [21, 28, 30, 31], efforts around this problem still continues. In complex Finsler geometry, there is also notions of complex Berwald metric and complex Landsberg metric, every complex Berwald metric is complex Landsberg metric [2], but the converse problem has been explored.

Our approach to this problem depends on the existence of solutions for system (6.22). If there was any, Corollary 6.8 and Corollary 6.12 provide us a method to construct complex non-Berwald Landsberg manifolds, as follows.

Let M_1 and M_2 be two complex Berwald manifolds, of course, they are complex Landsberg manifolds. Let λ_1 and λ_2 be positive smooth functions on product manifold $M_1 \times M_2$, and verifies (6.22), but they are not constants on $M_1 \times M_2$. According to Corollary 6.12, $(M_1 \times_{(\lambda_1,\lambda_2)} M_2, F)$ is a complex Landsberg manifold, while, the Corollary 6.8 guarantees that $(M_1 \times_{(\lambda_1,\lambda_2)} M_2, F)$ is not a Berwald manifold.

Our work takes a step forward on finding complex non-Berwald Landsberg manifolds. Investigating and characterizing possible solutions for system (6.22) will be the subject matter of our future works.

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