Mean Field Equations for the Equilibrium Turbulence and Toda Systems on Connected Finite Graphs

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Abstract. In this paper, we study existence of solutions of mean field equations for the equilibrium turbulence and Toda systems on connected finite graphs. Our method is based on calculus of variations, which was built on connected finite graphs by Grigor’yan, Lin and Yang.

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1 Introduction

Let $G = (V, E)$ be a connected finite graph, where $V$ is the vertex set and $E$ the edge set. For every edge $xy \in E$, we assume its weight $w_{xy} > 0$ and $w_{xy} = w_{yx}$. Denote by $\mu$ a positive and finite measure on $V$. Let us review some definitions first.

**Definition 1.1.** For any function $f$ on $V$, the $\mu$-Laplacian of $f$ is defined by

$$
\Delta_\mu f(x) = \frac{1}{\mu(x)} \sum_{y \sim x} w_{xy} [f(y) - f(x)],
$$

where $y \sim x$ means $xy \in E$. The associated gradient form is

$$
\Gamma(f, g)(x) = \frac{1}{2\mu(x)} \sum_{y \sim x} w_{xy} [f(y) - f(x)][g(y) - g(x)].
$$

The length of its gradient means

$$
|\nabla f|(x) = \sqrt{\Gamma(f, f)(x)}.
$$

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Now we are prepared to state our main results.

**Theorem 1.1.** Let \( G = (V, E) \) be a connected finite graph, \( \mu \) a positive and finite measure on \( V \). Let \( \psi : V \to \mathbb{R} \) be a function with \( \int_V \psi \, d\mu = 1 \). For \( i = 1, 2 \), we assume \( c_i > 0 \) is a constant and \( 0 \leq h_i \neq 0 \) is a function on \( V \). Then the mean field equation for the equilibrium turbulence

\[
\begin{cases}
\Delta \mu u = c_1 \left( \psi - \frac{h_1 e^u}{\int_V h_1 e^u \, d\mu} \right) - c_2 \left( \psi - \frac{h_2 e^{-u}}{\int_V h_2 e^{-u} \, d\mu} \right), \\
\int_V \psi u \, d\mu = 0
\end{cases}
\] (1.1)

has a solution on \( V \).

Our second result is about Toda system. We shall prove that

**Theorem 1.2.** Let \( G = (V, E) \) be a connected finite graph, \( \mu \) a positive and finite measure on \( V \). Let \( \psi : V \to \mathbb{R} \) be a function with \( \int_V \psi \, d\mu = 1 \). For \( i = 1, 2 \), we assume \( \lambda_i > 0 \) is a constant and \( 0 \leq \rho_i \neq 0 \) is a function on \( V \). Then the Toda system

\[
\begin{cases}
\Delta \mu u_1 = 2\lambda_1 \left( \psi - \frac{\rho_1 e^{u_1}}{\int_V \rho_1 e^{u_1} \, d\mu} \right) - \lambda_2 \left( \psi - \frac{\rho_2 e^{u_2}}{\int_V \rho_2 e^{u_2} \, d\mu} \right), \\
\Delta \mu u_2 = 2\lambda_2 \left( \psi - \frac{\rho_2 e^{u_2}}{\int_V \rho_2 e^{u_2} \, d\mu} \right) - \lambda_1 \left( \psi - \frac{\rho_1 e^{u_1}}{\int_V \rho_1 e^{u_1} \, d\mu} \right), \\
\int_V \psi u_i \, d\mu = 0, \quad i = 1, 2
\end{cases}
\] (1.2)

has a solution on \( V \).

On a compact Riemann surface \( (\Sigma, g) \), Eq. (1.1) describes the mean field of the equilibrium turbulence with arbitrarily signed vortices [1–3] and was obtained in [4, 5] from different statistical arguments. When \( \psi = 1/|\Sigma| \), Ohtsuka and Suzuki [6], using a variational argument, proved that Eq. (1.1) can be solved if \( 0 \leq c_1, c_2 < 8\pi \) and \( h_1 = h_2 = 1 \); Zhou [7, 8] gave a sufficient condition for the existence of solutions of Eq. (1.1) when \( c_1 = c_2 = 8\pi \) and \( h_1 = h_2 = 1 \); moreover, she studied the supercritical case of the existence of solutions of (1.1). These results partially generalized the existence results about Kazdan-Warner problem in [9].

On a compact Riemann surface \( (\Sigma, g) \), Eq. (1.2) is related to non-Abelian Chern-Simons model [10]. When \( \psi = 1/|\Sigma| \), Jost-Wang [11] proved the Moser-Trudinger inequality for Toda system. Based on this inequality, Li-Li [12] and Jost-Lin-Wang [13] gave a sufficient condition for the existence of solutions of the Toda system with \( \lambda_1 = \lambda_2 = 4\pi \), which is the critical case in the sense of the Moser-Trudinger inequality. For more relevant study, we refer the interesting reader to the references therein.

Recently, in a series of papers, Grigor’yan-Lin-Yang [14–16] founded the variational method for semi-linear elliptic equations on connected finite graphs and locally finite
graphs. Based on this method, many mathematicians studied equations with variational structure on graphs, see for example [17–25] and the references therein.

In this paper, motivated by [14], we are interested in the existence of solutions of mean field equations for the equilibrium turbulence and Toda systems on connected finite graphs. We shall pursue the variational method. We remark that, the existence of solutions of Eq. (1.1) on connected finite graphs and on compact Riemann surfaces are different. In connected finite graph case, Eq. (1.1) always has minimum solution if \( c_1, c_2 > 0 \), however, in compact Riemann surface case, Eq. (1.1) does not have minimum solution if \( c_1 \) or \( c_2 \) is bigger than \( 8\pi \). Similarly, in connected finite graph case, Toda system (1.2) always has a minimum solution if \( \lambda_1, \lambda_2 > 0 \), but in compact Riemann surface case, Toda system (1.2) does not have minimum solution when \( \lambda_1 \) or \( \lambda_2 \) is bigger than \( 4\pi \).

The rest of this paper is organized as follows: In Section 2, we present some well known knowledge about connected finite graphs. Then we prove our Theorems 1.1 and 1.2 in Sections 3 and 4 respectively. Throughout this paper, we use \( C \) to denote absolute constants without distinguishing them even in the same line; we do not distinguish sequence and its subsequence.

2 Preliminaries

In this section, we shall present some well known knowledge which are needed in the proofs of our main results, they come from [14] directly. Still, we shall give some short proofs for the convenience of readers.

Let \( G = (V, E) \) be a connected finite graph. Suppose \( u : V \to \mathbb{R} \) is a function, we define a norm

\[
\|u\| = \left( \int_V (|\nabla u|^2 + u^2) \, d\mu \right)^{1/2}.
\]

Under this norm, one has a Sobolev space

\[ W^{1,2}(V) = \{ u : u \text{ is a function on } V \text{ and } \|u\| < +\infty \}. \]

Since \( |V| \) is finite, we have \( W^{1,2}(V) = \mathbb{R}^{|V|} \). Then \( W^{1,2}(V) \) is pre-compact, that is,

**Lemma 2.1.** If \( \{u_j\} \) is bounded in \( W^{1,2}(V) \), then there exists some \( u_0 \in W^{1,2}(V) \) such that up to a subsequence, \( u_j \to u_0 \) in \( W^{1,2}(V) \) as \( j \to \infty \).

Still, we have the following inequalities of Poincaré and Moser-Trudinger.

**Lemma 2.2 (Poincaré inequality).** Suppose \( \psi : V \to \mathbb{R} \) is a function with \( \int_V \psi \, d\mu = 1 \). For every function \( u \) on \( V \) with \( \int_V \psi u \, d\mu = 0 \), there exists some constant \( C \) depending only on \( G \) such that

\[
\int_V u^2 \, d\mu \leq C \int_V |\nabla u|^2 \, d\mu.
\]
Proof. Suppose not. Then there exists a sequence of \( \{ u_n \} \subset W^{1,2}(V) \) with \( \int_V \psi u_n d\mu = 0 \) and \( \int_V u_n^2 d\mu = 1 \) and
\[
\lim_{n \to \infty} \int_V |\nabla u_n|^2 d\mu = 0.
\]
Since \( \{ u_n \} \) is bounded in \( W^{1,2}(V) \), from Lemma 2.1 we know, there is a subsequence of \( \{ u_n \} \) (which still denotes as \( \{ u_n \} \) for simplicity) and a function \( u_0 \in W^{1,2}(V) \) such that
\[
u_n \to u_0 \text{ in } W^{1,2}(V) \text{ as } n \to \infty.
\]
Thereby,
\[
\int_V \psi u_0 d\mu = \lim_{n \to \infty} \int_V \psi u_n d\mu = 0;
\]
\[
\int_V u_0^2 d\mu = \lim_{n \to \infty} \int_V u_n^2 d\mu = 1;
\]
\[
\int_V |\nabla u_0|^2 d\mu = \lim_{n \to \infty} \int_V |\nabla u_n|^2 d\mu = 0.
\]
Then the third and the first equalities tell us \( u_0 \equiv 0 \), which contradicts the second equality. This ends the proof of the lemma.

Lemma 2.3 (Moser-Trudinger inequality). Suppose \( \psi : V \to \mathbb{R} \) is a function with \( \int_V \psi d\mu = 1 \). For any \( \alpha > 0 \), there holds
\[
\sup_{u \in W^{1,2}(V), \int_V \psi u d\mu = 0, \int_V |\nabla u|^2 d\mu \leq 1} \int_V e^{\alpha u^2} d\mu < +\infty.
\]
Proof. For any \( u \in W^{1,2}(V) \) with \( \int_V \psi u d\mu = 0 \) and \( \int_V |\nabla u|^2 d\mu \leq 1 \), by Lemma 2.2 one knows
\[
\int_V u^2 d\mu \leq C \int_V |\nabla u|^2 d\mu \leq C.
\]
Then we have
\[
u^2(x) \leq \frac{1}{\min_{x \in V} \mu(x)} \int_V u^2 d\mu \leq C, \quad \forall u \in V.
\]
Therefore,
\[
\int_V e^{\alpha u^2} d\mu = \sum_{x \in V} \mu(x) e^{\alpha u^2(x)} \leq C \sum_{x \in V} \mu(x) \leq C.
\]
This proves the lemma.
3 Proof of Theorem 1.1

Consider the functional

\[ J(u) = \frac{1}{2} \int_V |\nabla u|^2 d\mu - c_1 \ln \int_V h_1 e^u d\mu - c_2 \ln \int_V h_2 e^{-u} d\mu, \quad u \in \mathcal{H}, \]

where \( \mathcal{H} \) is a closed nonempty subspace of \( W^{1,2}(V) \) defined by

\[ \mathcal{H} = \left\{ u \in W^{1,2}(V) : \int_V \psi u d\mu = 0 \right\}. \]

Using Cauchy’s inequality, one has for any \( \epsilon_i > 0 \) \((i = 1, 2)\) that

\[ u \leq \epsilon_1 \int_V |\nabla u|^2 d\mu + \frac{1}{4\epsilon_1} \left( \frac{u}{(\int_V |\nabla u|^2 d\mu)^{1/2}} \right)^2; \]

\[ -u \leq \epsilon_2 \int_V |\nabla u|^2 d\mu + \frac{1}{4\epsilon_2} \left( \frac{-u}{(\int_V |\nabla u|^2 d\mu)^{1/2}} \right)^2. \]

Combining with the Moser-Trudinger inequality (Lemma 2.3), one obtains

\[ \ln \int_V h_1 e^u d\mu \leq \ln \max_V h_1 + \ln \int_V e^{\epsilon_1 \int_V |\nabla u|^2 d\mu + \frac{1}{4\epsilon_1} \left( \frac{u}{(\int_V |\nabla u|^2 d\mu)^{1/2}} \right)^2} d\mu \leq C + \epsilon_1 \int_V |\nabla u|^2 d\mu \quad (3.1) \]

and

\[ \ln \int_V h_2 e^{-u} d\mu \leq \ln \max_V h_2 + \ln \int_V e^{\epsilon_2 \int_V |\nabla u|^2 d\mu + \frac{1}{4\epsilon_2} \left( \frac{-u}{(\int_V |\nabla u|^2 d\mu)^{1/2}} \right)^2} d\mu \leq C + \epsilon_2 \int_V |\nabla u|^2 d\mu. \quad (3.2) \]

Substituting (3.1) and (3.2) into the representation of \( J \), we obtain for \( u \in \mathcal{H} \) that

\[ J(u) \geq \left( \frac{1}{2} - c_1 \epsilon_1 - c_2 \epsilon_2 \right) \int_V |\nabla u|^2 d\mu - C \geq \frac{1}{3} \int_V |\nabla u|^2 d\mu - C \quad (3.3) \]

for sufficiently small \( \epsilon_i > 0 \) \((i = 1, 2)\). In view of (3.3), one knows \( J \) is bounded from below on \( \mathcal{H} \). So there exists a sequence \( \{u_n\} \subset \mathcal{H} \) such that

\[ \lim_{n \to \infty} J(u_n) = \inf_{u \in \mathcal{H}} J(u). \quad (3.4) \]

For any \( \epsilon > 0 \), one has by combining (3.3) and (3.4) that

\[ \frac{1}{3} \int_V |\nabla u_n|^2 d\mu - C \leq J(u_n) \leq \inf_{u \in \mathcal{H}} J(u) + \epsilon \leq J(0) + \epsilon \leq C, \quad (3.5) \]
when \( n \) is sufficiently large. From (3.5) and the Poincaré inequality (Lemma 2.2), we have \( \{u_n\} \) is bounded in \( W^{1,2}(V) \). By Lemma 2.1, there is a function \( u_0 \in W^{1,2}(V) \) such that up to a subsequence

\[
u_n \to u_0 \text{ in } W^{1,2}(V) \text{ as } n \to \infty. \tag{3.6}\]

It then follows by Hölder’s inequality that

\[
\left| \int_{V} \psi(u_n - u_0) \, d\mu \right| \leq \max_{V} |\psi| \left( \int_{V} |u_n - u_0|^2 \, d\mu \right)^{1/2} \mu(V)^{1/2} \to 0 \text{ as } n \to \infty, \tag{3.7}
\]

and therefore \( \int_{V} \psi u_0 \, d\mu = 0 \) and \( u_0 \in \mathcal{H} \).

Calculating directly, one has

\[
\left| \int_{V} h_1(e^{u_n} - e^{u_0}) \, d\mu \right| = \left| \int_{0}^{1} \frac{d}{dt} \int_{V} h_1 e^{t(u_n - u_0) + u_0} (u_n - u_0) \, d\mu \right| = \max_{V} h_1 \int_{0}^{1} \left( \int_{V} e^{2t(u_n - u_0) + 2u_0} \, d\mu \right)^{1/2} \left( \int_{V} |u_n - u_0|^2 \, d\mu \right)^{1/2} \, dt
\]

\[
\leq \max_{V} h_1 \int_{0}^{1} \left( \int_{V} e^{\left( \frac{t(u_n - u_0) + u_0}{v_0\rho_{2,2}(V)} \right)^2 + 4\|\nabla(u_n - u_0) + \nabla u_0\|^2_{2(V)} \, d\mu \right)^{1/2} \left( \int_{V} |u_n - u_0|^2 \, d\mu \right)^{1/2} \, dt
\]

\[
\leq C e^{4\left( \|\nabla u_n\|^2_{2(V)} + \|\nabla u_0\|^2_{2(V)} \right)} \left( \int_{V} |u_n - u_0|^2 \, d\mu \right)^{1/2} \to 0 \text{ as } n \to \infty, \tag{3.8}
\]

where the Moser-Trudinger inequality (Lemma 2.3) and (3.6) were used. Similarly, one has

\[
\int_{V} h_2(e^{-u_0} - e^{-u_0}) \, d\mu \to 0 \text{ as } n \to \infty. \tag{3.9}
\]

Combining (3.6), (3.8) and (3.9), one obtains

\[
J(u_0) = \lim_{n \to \infty} J(u_n) = \inf_{u \in \mathcal{H}} J(u).
\]

Finally, by a direct variational method we have \( u_0 \) satisfies the Euler-Lagrange equation (1.1).

## 4 Proof of Theorem 1.2

Consider the functional

\[
I(u_1, u_2) = \frac{1}{3} \int_{V} \left( |\nabla u_1|^2 + \nabla u_1 \cdot \nabla u_2 + |\nabla u_2|^2 \right) \, d\mu - \lambda_1 \ln \int_{V} \rho_1 e^{u_1} \, d\mu - \lambda_2 \ln \int_{V} \rho_2 e^{u_2} \, d\mu,
\]
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where $u_1, u_2 \in \mathcal{H}$. For $i = 1, 2$, let $\epsilon_i > 0$ be chosen later. Similarly as (3.1), one can easily prove that
\[
\ln \int_V \rho_i e^{u_i} d\mu \leq C + \epsilon_i \int_V |\nabla u_i|^2 d\mu.
\] (4.1)

By (4.1), we have
\[
I(u_1, u_2) \geq \left( \frac{1}{6} - \lambda_1 \epsilon_1 \right) \int_V |\nabla u_1|^2 d\mu + \left( \frac{1}{6} - \lambda_2 \epsilon_2 \right) \int_V |\nabla u_2|^2 d\mu - C
\] \[
\geq \frac{1}{8} \int_V (|\nabla u_1|^2 + |\nabla u_2|^2) d\mu - C,
\] (4.2)

where one can choose $\epsilon_i = 1/24 \lambda_i$ for $i = 1, 2$. It follows from (4.2) that $I(u_1, u_2)$ in $\mathcal{H} \times \mathcal{H}$ is bounded from below. So there exists a sequence \( \{(u^n_1, u^n_2) \} \subset \mathcal{H} \times \mathcal{H} \) such that
\[
\lim_{n \to \infty} I(u^n_1, u^n_2) = \inf_{\mathcal{H} \times \mathcal{H}} I(u_1, u_2).
\] (4.3)

For any $\epsilon > 0$, we have by (4.2) and (4.3) that
\[
\frac{1}{8} \int_V (|\nabla u^n_1|^2 + |\nabla u^n_2|^2) d\mu - C \leq I(u^n_1, u^n_2) \leq \inf_{\mathcal{H} \times \mathcal{H}} I(u_1, u_2) + \epsilon \leq I(0, 0) + \epsilon \leq C
\] (4.4)

when $n$ is sufficiently large. From (4.4) and the Poincaré inequality in Lemma 2.2, one knows $u^n_1$ and $u^n_2$ are bounded in $W^{1,2}(V)$. Similarly as (3.6), (3.7) and (3.8), there exists a $(u^0_1, u^0_2) \in \mathcal{H} \times \mathcal{H}$ such that
\[
I(u^0_1, u^0_2) = \lim_{n \to \infty} I(u^n_1, u^n_2) = \inf_{\mathcal{H} \times \mathcal{H}} I(u_1, u_2).
\]

Therefore, $(u^0_1, u^0_2)$ satisfies the Euler-Lagrange equation (1.2). This ends the proof of Theorem 1.2.

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**References**


For journal names, follow “Abbreviations of Names of Serials” reviewed in Mathematical Reviews. Cite references in the text by Arabic number between square brackets, as [1], [1, 2], [1, Theorem 1.1], etc.