Two-Step Two-Sweep Modulus-Based Matrix Splitting Iteration Method for Linear Complementarity Problems

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Abstract. Linear complementarity problems have drawn considerable attention in recent years due to their wide applications. In this article, we introduce the two-step two-sweep modulus-based matrix splitting (TSTM) iteration method and two-sweep modulus-based matrix splitting type II (TM II) iteration method which are a combination of the two-step modulus-based method and the two-sweep modulus-based method, as two more effective ways to solve the linear complementarity problems. The convergence behavior of these methods is discussed when the system matrix is either a positive-definite or an $H_+$-matrix. Finally, numerical experiments are given to show the efficiency of our proposed methods.

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Key words: Linear complementarity problem, modulus-based method, two-step, two sweep, $H_+$-matrix, convergence.

1. Introduction

Linear complementarity problems attract many researchers' attention as a current research field. These problems arise typically in various practical applications from economics, engineering, and sciences. The study of numerical techniques to solve these problems is significant accordingly.

For the given pair of vector $q \in \mathbb{R}^n$ and matrix $A \in \mathbb{R}^{n \times n}$, linear complementarity problem (LCP($q, A$)) is to find a vector $z \in \mathbb{R}^n$ such that

$$z \geq 0, \quad \omega := Az + q \geq 0, \quad z^T \omega = 0. \quad (1.1)$$

LCP($q, A$) commonly involves optimization strategies such as linear or quadratic programming, optimal capital invariant stock, optimal stopping problems, and many more [7–12, 17, 23, 25, 29, 32, 43, 44, 48].

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Direct and iterative methods are two major categories of numerical methods which widely use for solving LCPs\((q, A)\) \([1, 3, 6, 13, 14, 21, 22, 28, 30, 38, 46]\). Among them, modulus-based matrix splitting iteration methods have a special place \([31, 37]\). In 1981, Van Bokhoven \([38]\) presented a modulus iteration method by reformulating the LCP\((q, A)\) as an implicit fixed point equation for the first time. Afterward, based on the modulus iteration method \([38]\) and the matrix splitting of the system matrix \(A\), Bai \([2]\) developed LCP\((q, A)\) that led to a series of other modulus-based matrix splitting iteration methods by presenting a general framework of modulus-based matrix splitting iteration methods. In this regard, various modulus-based matrix splitting iteration methods have been proposed \([4, 15, 16, 27, 43, 45, 50, 52, 53, 56, 57]\). Convergence analysis of these methods for solving LCP\((q, A)\) was discussed when the system matrix is either a positive-definite or an \(H_+\)-matrix.

In 2011, Zhang \([49]\) suggested the two-step modulus-based matrix splitting iteration method and discussed its convergence, when the system matrix is an \(H_+\)-matrix and \(A = M_1 - N_1 = M_2 - N_2\) are \(H\)-compatible splittings. Afterward, in 2014, Ke and Ma \([24]\) established new convergence conditions. To improve the convergence conditions presented in \([49]\), Zhang \([47]\) assumed \(\Omega\) satisfy \(\Omega \geq D_A\) for the positive diagonal matrix \(\Omega\). Among the written articles in the field of the two-step modulus-based matrix splitting iteration method, we can mention \([26, 47, 49, 54]\).

In recent years, the two-sweep modulus-based method has been studied as another effective module-based method for solving LCP\((q, A)\). For the first time, in 2016, by using the identical equation, the two-sweep modulus-based matrix splitting iteration method was extended to LCP\((q, A)\) \([42]\). This method can also yield a series of two-sweep modulus-based matrix splitting iteration methods, such as two-sweep modulus-based Jacobi, two-sweep modulus-based Gauss-Seidel, etc. By putting another diagonal matrix parameter to the fixed point iteration formula, a general two-sweep modulus-based matrix splitting iteration method for LCP\((q, A)\) was established. Also, Peng \([33]\) introduced a relaxation two-sweep modulus-based matrix splitting iteration method for LCP\((q, A)\) based on choosing different pairs of relaxation parameters. In this article, we use a combination of two-sweep and two-step modulus-based matrix splitting iteration methods to introduce two classes of more efficient methods for solving LCPs\((q, A)\). The convergence analysis of these methods is studied under suitable conditions. This paper consists of the following section.

Section 2 contains necessary items that we need throughout the article. Section 3 presents a two-step two-sweep modulus-based matrix splitting iteration method and a two-sweep modulus-based matrix splitting type II iteration method for LCP\((q, A)\). Section 4 discusses the convergence analysis. Section 5 uses three numerical examples to illustrate that the presented methods are feasible. Finally, we present conclusions.

2. Preliminaries

In this section, some of the necessary definitions, notations, and lemmas are briefly introduced, see \([1, 33, 36, 39, 41]\). For \(A = (a_{ij}) \in \mathbb{R}^{n \times n}\), \(A\) is called nonnegative
Furthermore, a matrix \( A \) is called

- a Z-matrix if for all \( i \neq j \), \( a_{ij} \leq 0 \),
- an L-matrix if \( a_{ii} > 0 \), \( i = 1, \ldots, n \) and \( a_{ij} \leq 0 \), \( i \neq j \), \( i, j = 1, 2, \ldots, n \),
- an M-matrix if \( A \) is a Z-matrix and nonsingular with \( A^{-1} \geq 0 \),
- an H-matrix, if its comparison matrix \( \langle A \rangle \) is an M-matrix, where the comparison matrix \( \langle A \rangle = (a_{ij}) \) for \( A \) is defined by

\[
\langle A \rangle = (\langle a_{ij} \rangle) = \begin{cases} |a_{ij}|, & i = j, \\ -|a_{ij}|, & i \neq j. \end{cases} \tag{2.1}
\]

Also, \( A \) is an \( H_+ \)-matrix, when \( A \) is an \( H \)-matrix with positive diagonal entries [1, 5]. Furthermore, a matrix \( A \) is called a \( P \)-matrix if all its principal minors are positive. It turns that a matrix \( A \) is a \( P \)-matrix if and only if the corresponding LCP(\( q, A \)) has a unique solution for any \( q \in \mathbb{R}^n \). We note that any \( H_+ \)-matrix and any positive-definite matrix are \( P \)-matrices [10].

According to [33, 41] the decomposition \( A = M - N \) is called:

- Convergent splitting of \( A \), if \( A \) and \( M \) are nonsingular matrices and \( \rho(M^{-1}N) < 1 \).
- \( M \)-splitting of \( A \), if \( M \) is a nonsingular \( M \)-matrix and \( N \geq 0 \).
- \( H \)-splitting of \( A \), if \( \langle M \rangle - |N| \) is an \( M \)-matrix.
- \( H \)-compatible splitting of \( A \), if \( \langle A \rangle = \langle M \rangle - |N| \), \( (H \)-splitting\).

Let us recall that: For an arbitrary \( H \)-matrix \( A \), that \( A \) is nonsingular \( |A^{-1}| \leq \langle A \rangle^{-1} \).

**Lemma 2.1** ([34]). If

\[
W = \begin{bmatrix} E & F \\ I & 0 \end{bmatrix} \geq 0, \tag{2.2}
\]

and \( \rho(E + F) < 1 \), then \( \rho(W) < 1 \).

**Lemma 2.2** ([18]). Let \( A \) be an \( H \)-matrix and \( A = D - B \), where \( D \) is the diagonal part of the matrix \( A \). Then the following statements hold true:

(i) \( A \) is nonsingular and \( |A^{-1}| \leq \langle A \rangle^{-1} \),

(ii) \( |D| \) is nonsingular and \( \rho(|D|^{-1}B) \) \( < 1 \).
Lemma 2.3 ([5]). Let $A$ be a $Z$-matrix. Then the following statements are equivalent:

- $A$ is an $M$-matrix.
- There exists a positive diagonal matrix $G$, such that $AG$ is an s.d.d. matrix with positive diagonal entries.
- There exists a positive vector $x$, such that $Ax > 0$.
- For any splitting $A = M - N$ of $A$ satisfying $M^{-1} \geq 0$ and $N \geq 0$, it holds $ho(M^{-1}N) < 1$.

Lemma 2.4 ([2]). Let $A = M - N$ be a splitting of the matrix $A \in \mathbb{R}^{n \times n}$, $\Omega \in \mathbb{R}^{n \times n}$ be a positive diagonal matrix, and $\gamma$ be a positive constant. For the LCP($q, A$), the following statements hold true:

(i) If $(z, \omega)$ is a solution of the LCP($q, A$), then

$$x = \frac{1}{2} \gamma (z - \Omega^{-1}\omega)$$

with

$$|x| = \frac{1}{2} \gamma (z + \Omega^{-1}\omega)$$

satisfies the implicit fixed-point equation

$$(\Omega + M)x = Nx + (\Omega - A)|x| - \gamma q.$$ (2.3)

(ii) If $x$ satisfies the implicit fixed-point equation (2.3), then

$$z = \gamma^{-1}(|x| + x), \quad \omega = \gamma^{-1}\Omega(|x| - x)$$

is a solution of the LCP($q, A$).

Lemma 2.5 ([24]). For a nonnegative matrix $A \in \mathbb{R}^{n \times n}$, if there exists a positive vector $v \in \mathbb{R}^n$ such that $Av < v$, then $\rho(A) < 1$.

Lemma 2.6 ([19]). Let $A, B \in \mathbb{R}^{n \times n}$ such that $\langle A \rangle \leq \langle B \rangle$. If $A$ is an $H$-matrix, then $B$ is an $H$-matrix as well.

Theorem 2.1 ([20]). Let $A = M - N$ be a splitting.

(a) If the splitting is regular or weak regular, then $\rho(M^{-1}N) < 1$ if and only if $A^{-1} \geq 0$.

(b) If the splitting is an $M$-splitting, then $\rho(M^{-1}N) < 1$ if and only if $A$ is an $M$-matrix.

(c) If the splitting is an $H$-splitting, then $A$ and $M$ are $H$-matrices and $\rho(M^{-1}N) \leq \rho(\langle M^{-1} \rangle|N|) < 1$.

(d) If the splitting is an $M$-splitting, then it is a regular splitting.

(e) If the splitting is an $M$-splitting and $A$ is an $M$-matrix, then it is an $H$-splitting and also an $H$-compatible splitting.

(f) If the splitting is an $H$-compatible splitting and $A$ is an $H$-matrix, then it is an $H$-splitting and thus convergent.
3. Derivation of methods for solving LCP \((q, A)\)

In this section, for solving LCP \((q, A)\), firstly, the two-step modulus-based matrix splitting iteration method and the two-sweep modulus-based matrix splitting iteration method are described as the basis of our proposed methods. Afterward, the proposed methods are introduced in detail. Let \(Ω\) be a positive diagonal matrix and \(γ\) be a positive constant, for solving the LCP \((q, A)\), by setting \(z = \left(\frac{1}{γ}\right)(|x| + x)\), \(ω = \left(\frac{1}{γ}\right)Ω(|x| − x)\)

based on (2.3), the following modulus-based matrix splitting iteration method presented [2],

\[
Ω + M\right)^{(k+1)}_x = N_1\right)^{(k)}_x + (Ω − A)|x^{(k)}| − γq.
\]

In this regard, based on Eq. (3.1) Zhang [49] proposed the two-step modulus-based matrix splitting iteration method as follows.

**Method 3.1** Two-step modulus-based matrix splitting iteration method for solving LCP \((q, A)\) [49].

1. Given an initial vector \(x^{(0)}\) ∈ \(R^n\), and set \(k = 0\).
2. Compute \(x^{(k+1)}\) ∈ \(R^n\) by solving the systems

\[
\begin{cases}
(Ω + M_1)x^{(k+\frac{1}{2})} = N_1\right)^{(k)}_x + (Ω − A)|x^{(k)}| − γq, \\
(Ω + M_2)x^{(k+1)} = N_2\right)^{(k+\frac{1}{2})}_x + (Ω − A)|x^{(k+\frac{1}{2})}| − γq.
\end{cases}
\]

3. Set

\[
z^{(k+1)} = \left(\frac{1}{γ}\right)\left(|x^{(k+1)}| + x^{(k+1)}\right).
\]

4. If \(z^{(k+1)}\) satisfies the stopping rule, then stop. Otherwise, set \(k = k + 1\) and return to Step 2.

On the other hand, based on (3.1), the following two-sweep modulus-based matrix splitting iteration method was set up in [42].

**Method 3.2** Two-sweep modulus-based matrix splitting iteration method for solving LCP \((q, A)\) [42].

Let \(A = M − N\) be a splitting of matrix \(A \in R^{n \times n}\).

1. Given two initial vectors \(x^{(0)}\), \(x^{(1)}\) ∈ \(R^n\) and set \(k = 1\).
2. Compute \(x^{(k+1)}\) ∈ \(R^n\) by solving the linear system

\[
(Ω + M)x^{(k+1)} = N\right)^{(k)}_x + (Ω − A)|x^{(k−1)}| − γq.
\]
As two-step modulus-based and two-sweep modulus-based methods are two significant and effective methods to decrease the CPU time and iteration steps for some LCPs \((q, A)\), to achieve higher computing efficiency, we establish the two-step two-sweep modulus-based matrix splitting (TSTM) iteration methods for the LCP \((q, A)\).

**Method 3.3** The TSTM iteration method for solving LCP \((q, A)\).

Let \(A = M_1 - N_1 = M_2 - N_2\) be two splittings of matrix \(A \in \mathbb{R}^{n \times n}\).

1. Given two initial vectors \(x^{(0)}, x^{(1)} \in \mathbb{R}^n\) and set \(k = 1\).
2. Compute \(x^{(k+1)} \in \mathbb{R}^n\) by solving the system
   \[
   \begin{align*}
   (\Omega + M_2)x^{(k+\frac{1}{2})} &= N_2x^{(k)} + (\Omega - A)|x^{(k-1)}| - \gamma q, \\
   (\Omega + M_1)x^{(k+1)} &= N_1x^{(k+\frac{1}{2})} + (\Omega - A)|x^{(k)}| - \gamma q.
   \end{align*}
   \]
3. Set
   \[
   z^{(k+1)} = \left(\frac{1}{\gamma}\right) \left( |x^{(k+1)}| + x^{(k+1)} \right).
   \]
4. If \(z^{(k+1)}\) satisfies the stopping rule, then terminate. Otherwise, set \(k = k + 1\) and return to Step 2.

In the following, we introduce two-sweep modulus-based matrix splitting type II (TM II) iteration method for solving LCP \((q, A)\). Numerical results show that this method works better than the two-step two-sweep modulus-based matrix splitting iteration method in real implementation.

**Method 3.4** The TM II iteration method for solving LCP \((q, A)\).

Let \(A = M_1 - N_1 = M_2 - N_2\) be two splittings of matrix \(A \in \mathbb{R}^{n \times n}\).

1. Given two initial vectors \(x^{(0)}, x^{(1)} \in \mathbb{R}^n\) and set \(k = 1\).
2. Compute \(x^{(k+1)} \in \mathbb{R}^n\) by solving the system
   \[
   \begin{align*}
   (\Omega + M_2)x^{(k+\frac{1}{2})} &= N_2x^{(k)} + (\Omega - A)|x^{(k-1)}| - \gamma q, \\
   (\Omega + M_1)x^{(k+1)} &= N_1x^{(k+\frac{1}{2})} + (\Omega - A)|x^{(k)}| - \gamma q.
   \end{align*}
   \]
3. Set
   \[
   z^{(k+1)} = \left(\frac{1}{\gamma}\right) \left( |x^{(k+1)}| + x^{(k+1)} \right).
   \]
4: If \( z^{(k+1)} \) satisfies the stopping rule, then stop. Otherwise, set \( k = k + 1 \) and return to Step 2.

**Remark 3.1.** Varying choices of matrix splitting can lead to different modulus-based matrix splitting iteration methods. Methods 3.3 and 3.4 provide two general frameworks of two-step two-sweep modulus-based matrix splitting and two-sweep modulus-based matrix splitting type II for solving \( LCP(q, A) \). For instance, assume that \( A \in \mathbb{R}^{n \times n} \) is an \( H_+ \)-matrix, and \( A = D - L - U = D - B \), where \( D, L, \) and \( U \) are diagonal, strictly lower-triangular, and strictly upper-triangular matrices of the matrix \( A \), respectively. Let \( A = M_1 - N_1 = M_2 - N_2 \) be two splittings of \( A \) where

\[
M_1 = \frac{1}{\alpha}(D - \beta L), \quad N_1 = \frac{1}{\alpha}((1 - \alpha)D + (\alpha - \beta)L + \alpha U),
\]

\[
M_2 = \frac{1}{\alpha}(D - \beta U), \quad N_2 = \frac{1}{\alpha}((1 - \alpha)D + (\alpha - \beta)U + \alpha L),
\]

and \( \gamma = 2 \). Method 3.3 gives the two-step two-sweep modulus-based accelerated overrelaxation (TSTMAOR) method as follows:

\[
\begin{align*}
\begin{cases}
(\alpha \Omega + D - \beta U)x^{(k+\frac{1}{2})} \\
= \left((1 - \alpha)D + (\alpha - \beta)U + \alpha L\right)x^{(k)} + \alpha(\Omega - A)|x^{(k-1)}| - 2\alpha q,
\end{cases}
\end{align*}
\]

with

\[
z^{(k+1)} = \frac{1}{2}\left(|x^{(k+1)}| + x^{(k+1)}\right).
\]

The TSTMAOR method reduces to the two-step two-sweep modulus-based successive overrelaxation (TSTMSOR) method, the two-step two-sweep modulus-based Gauss-Seidel (TSTMGS) method, the two-step two-sweep modulus-based Jacobi (TSTMJ) method when \( \alpha = \beta, \alpha = \beta = 1 \) and \( \alpha = 1, \beta = 0 \), respectively. Similarly, we have these items for Method 3.4, where the two-sweep modulus-based matrix splitting type II accelerated overrelaxation (TM II AOR) iteration method is as follows:

\[
\begin{align*}
\begin{cases}
(\alpha \Omega + D - \beta L)x^{(k+\frac{1}{2})} \\
= \left((1 - \alpha)D + (\alpha - \beta)L + \alpha U\right)x^{(k)} + \alpha(\Omega - A)|x^{(k-1)}| - 2\alpha q,
\end{cases}
\end{align*}
\]

with

\[
z^{(k+1)} = \frac{1}{2}\left(|x^{(k+1)}| + x^{(k+1)}\right).
\]
4. Convergence results

This section discusses the convergence of the TSTM and TM II iteration methods for solving LCP\((q, A)\) when the system matrix is either a positive-definite or an \(H_+\)-matrix.

**Theorem 4.1.** Suppose \(A \in \mathbb{R}^{n \times n}\) is an \(H_+\)-matrix, \(\Omega \in \mathbb{R}^{n \times n}\) is a positive diagonal matrix, \(\gamma\) is a positive constant and \(A = M_1 - N_1 = M_2 - N_2\) are two \(H\)-compatible splittings of \(A\). Assume the positive diagonal matrix \(\Omega\) satisfies \(\Omega \geq D\). Then, for any two initial vectors \(x^{(0)}\) and \(x^{(1)}\), the iteration sequence \(\{z^{(k)}\}_{k=0}^{\infty}\) generated by Method 3.3 converges to the unique solution \(z^* \in \mathbb{R}_+^n\) of the LCP\((q, A)\).

**Proof.** We assume that \(z^* \in \mathbb{R}^n\) is the exact solution of LCP\((q, A)\), then

\[
x^* = \left(\frac{1}{2}\right) (z^* - \Omega^{-1}w^*)
\]

with

\[
|x^*| = \left(\frac{1}{2}\right) (z^* + \Omega^{-1}w^*) ,
\]

satisfies the implicit fixed-point equations, see [49] for more details

\[
\begin{align*}
((\Omega + M_2)x^* &= N_2x^* + (\Omega - A)|x^*| - \gamma q, \\
(\Omega + M_1)x^* &= N_1x^* + (\Omega - A)|x^*| - \gamma q,
\end{align*}
\]

or

\[
\begin{bmatrix}
 x^* \\
 x^* \\
| x^* |
\end{bmatrix} =
\begin{bmatrix}
 I & 0 \\
 (\Omega + M_2)^{-1}N_2 & (\Omega + M_2)^{-1}(\Omega - A) \\
 (\Omega + M_1)^{-1}(\Omega - A) & (\Omega + M_1)^{-1}N_1 \\
 I & 0
\end{bmatrix}
\begin{bmatrix}
 x^* \\
|x^*| \\
 x^*
\end{bmatrix} +
\begin{bmatrix}
 0 \\
 -\gamma (\Omega + M_2)^{-1}q \\
 -\gamma (\Omega + M_1)^{-1}q \\
 0
\end{bmatrix}.
\]

From (3.4) and (4.1), we get

\[
\begin{align*}
x^{(k+\frac{1}{2})} &= (\Omega + M_2)^{-1}N_2x^{(k)} + (\Omega + M_2)^{-1}(\Omega - A)|x^{(k-1)}| - (\Omega + M_2)^{-1}\gamma q, \\
x^{(k+1)} &= (\Omega + M_1)^{-1}N_1x^{(k+\frac{1}{2})} + (\Omega + M_1)^{-1}(\Omega - A)|x^{(k)}| - (\Omega + M_1)^{-1}\gamma q,
\end{align*}
\]

or

\[
\begin{align*}
x^k &= x^k, \\
x^{(k+\frac{1}{2})} &= (\Omega + M_2)^{-1}N_2x^{(k)} + (\Omega + M_2)^{-1}(\Omega - A)|x^{(k-1)}| - (\Omega + M_2)^{-1}\gamma q, \\
x^{(k+1)} &= (\Omega + M_1)^{-1}N_1x^{(k+\frac{1}{2})} + (\Omega + M_1)^{-1}(\Omega - A)|x^{(k)}| - (\Omega + M_1)^{-1}\gamma q, \\
| x^k | &= | x^k |,
\end{align*}
\]
or

\[
\begin{bmatrix}
  x^{(k)} \\
  x^{(k+\frac{1}{2})} \\
  |x^k| - |x^*|
\end{bmatrix} = \begin{bmatrix}
  I & 0 & 0 \\
  (\Omega + M_2)^{-1}N_2 & (\Omega + M_2)^{-1}(\Omega - A) & -\gamma(\Omega + M_2)^{-1}q \\
  (\Omega + M_1)^{-1}(\Omega - A) & (\Omega + M_1)^{-1}N_1 & -\gamma(\Omega + M_1)^{-1}q
\end{bmatrix} \begin{bmatrix}
  x^{(k)} \\
  x^{(k+\frac{1}{2})} \\
  |x^k| - |x^*|
\end{bmatrix},
\]  

(4.6)

Based on (4.6) and (4.2), we have

\[
\begin{bmatrix}
  x^{(k)} - x^* \\
  x^{(k+\frac{1}{2})} - x^* \\
  |x^k| - |x^*|
\end{bmatrix} = \begin{bmatrix}
  I & 0 & 0 \\
  (\Omega + M_2)^{-1}N_2 & (\Omega + M_2)^{-1}(\Omega - A) & -\gamma(\Omega + M_2)^{-1}q \\
  (\Omega + M_1)^{-1}(\Omega - A) & (\Omega + M_1)^{-1}N_1 & -\gamma(\Omega + M_1)^{-1}q
\end{bmatrix} \begin{bmatrix}
  x^{(k)} - x^* \\
  x^{(k+\frac{1}{2})} - x^* \\
  |x^k| - |x^*|
\end{bmatrix},
\]  

(4.7)

Then

\[
\begin{bmatrix}
  x^{(k)} - x^* \\
  x^{(k+\frac{1}{2})} - x^* \\
  |x^k| - |x^*|
\end{bmatrix} = \begin{bmatrix}
  I & 0 & 0 \\
  (\Omega + M_2)^{-1}N_2 & (\Omega + M_2)^{-1}(\Omega - A) & -\gamma(\Omega + M_2)^{-1}q \\
  (\Omega + M_1)^{-1}(\Omega - A) & (\Omega + M_1)^{-1}N_1 & -\gamma(\Omega + M_1)^{-1}q
\end{bmatrix} \begin{bmatrix}
  x^{(k)} - x^* \\
  x^{(k+\frac{1}{2})} - x^* \\
  |x^k| - |x^*|
\end{bmatrix},
\]  

(4.8)

Since \( A = M_1 - N_1 = M_2 - N_2 \) are two \( H \)-compatible splittings of the \( H_+ \)-matrix \( A \). Based on Theorem 4.1, matrix \( M_1 \) and \( M_2 \) are \( H \)-matrices and we have

\[
\begin{cases}
  a_{ii} = m_{1ii} - n_{1ii} = m_{2ii} - n_{2ii}, & i = 1, 2, \ldots, n, \\
  |m_{1ii}| - |n_{1ii}| > 0, & |m_{2ii}| - |n_{2ii}| > 0.
\end{cases}
\]  

(4.9)

From (4.9), we obtain that \( m_{1ii} > 0 \) and \( m_{2ii} > 0 \). Therefore \( \text{diag}(M_1) > 0, \text{diag}(M_2) > 0 \) and \( \Omega + M_1, \Omega + M_2 \) are \( H_+ \)-matrices. According to Lemma 2.2,

\[
0 \leq \|(\Omega + M_1)^{-1}\| \leq (\Omega + (M_1)^{-1})^{-1},
\]

\[
0 \leq \|(\Omega + M_2)^{-1}\| \leq (\Omega + (M_2)^{-1})^{-1}.
\]  

(4.10)

By using the triangle inequality \( ||x^k| - |x^*|| \leq |x^k - x^*|, \) (4.8) and (4.10) we have

\[
\begin{bmatrix}
  |x^{(k+1)} - x^*| \\
  |x^{(k)} - x^*|
\end{bmatrix} = \begin{bmatrix}
  (\Omega + M_1)^{-1}(\Omega - A) & (\Omega + M_1)^{-1}N_1 \\
  I & 0
\end{bmatrix} \begin{bmatrix}
  |x^{(k)} - x^*| \\
  |x^{(k)} - x^*|
\end{bmatrix}.
\]
Assume that the unique solution generated by Method 3.3 converges and implies convergence of the sequence \( \{S_n\} \subset R^n \). If \( \rho(F) < 1 \), then for any two initial vectors, the iteration sequence \( \{x^{(k)}\} \subset R^n \) generated by Method 3.3 converges and implies convergence of the sequence \( \{z^{(k)}\} \subset R^n \) to the unique solution \( z^* \in R^n \) of the LCP \((q, A)\). Obviously, \( F \geq 0 \), based on Lemma 2.1, we only need to show that \( \rho(S) < 1 \), where

\[
S = (\langle M_1 \rangle)^{-1}(\Omega - A) + (\langle M_1 \rangle)^{-1}|N_1|(\langle M_2 \rangle)^{-1}|N_2|
+ (\langle M_1 \rangle)^{-1}|N_1|(\Omega + \langle M_2 \rangle)^{-1}|N_2|
+ (\langle M_1 \rangle)^{-1}|N_1|(\Omega + \langle M_2 \rangle)^{-1}|N_2|
+ (\langle M_1 \rangle)^{-1}(\langle M_2 \rangle)^{-1}|N_2| + (\langle M_2 \rangle)^{-1}|N_2|
+ (\langle M_1 \rangle)^{-1}(\langle M_2 \rangle)^{-1} + \langle M_2 \rangle - \langle M_2 \rangle + \Omega - \langle A \rangle
+ (\langle M_1 \rangle)^{-1}(\langle M_2 \rangle)^{-1}(\langle M_2 \rangle - \langle M_2 \rangle) + \Omega - \langle A \rangle
+ (\langle M_1 \rangle)^{-1}(\langle M_2 \rangle)^{-1}(\langle M_2 \rangle - \langle M_2 \rangle) + \Omega - \langle A \rangle
+ (\langle M_1 \rangle)^{-1}(\langle M_2 \rangle)^{-1}(\langle M_2 \rangle - \langle M_2 \rangle) + \Omega - \langle A \rangle
+ (\langle M_1 \rangle)^{-1}(\langle M_2 \rangle)^{-1}(\langle M_2 \rangle - \langle M_2 \rangle) + \Omega - \langle A \rangle
+ (\langle M_1 \rangle)^{-1}(\langle M_2 \rangle)^{-1}(\langle M_2 \rangle - \langle M_2 \rangle) + \Omega - \langle A \rangle
+ (\langle M_1 \rangle)^{-1}(\langle M_2 \rangle)^{-1}(\langle M_2 \rangle - \langle M_2 \rangle) + \Omega - \langle A \rangle
+ (\langle M_1 \rangle)^{-1}(\langle M_2 \rangle)^{-1}(\langle M_2 \rangle - \langle M_2 \rangle) + \Omega - \langle A \rangle
+ (\langle M_1 \rangle)^{-1}(\langle M_2 \rangle)^{-1}(\langle M_2 \rangle - \langle M_2 \rangle) + \Omega - \langle A \rangle
+ (\langle M_1 \rangle)^{-1}(\langle M_2 \rangle)^{-1}(\langle M_2 \rangle - \langle M_2 \rangle) + \Omega - \langle A \rangle
+ (\langle M_1 \rangle)^{-1}(\langle M_2 \rangle)^{-1}(\langle M_2 \rangle - \langle M_2 \rangle) + \Omega - \langle A \rangle

Assume that

\[
F = \begin{bmatrix} t_1 & t_2 \\ I & 0 \end{bmatrix},
\]

(4.12)

if \( \rho(F) < 1 \), then for any two initial vectors, the iteration sequence \( \{x^{(k)}\} \subset R^n \) generated by Method 3.3 converges and implies convergence of the sequence \( \{z^{(k)}\} \subset R^n \) to the unique solution \( z^* \in R^n \) of the LCP \((q, A)\). Obviously, \( F \geq 0 \), based on Lemma 2.1, we only need to show that \( \rho(S) < 1 \), where

\[
S = \begin{bmatrix} x^{(k+1)} - x^* \\ x^k - x^* \end{bmatrix}
\]

(4.11)
\[
\begin{align*}
&= I + (\Omega + \langle M_1 \rangle)^{-1} \left( |N_1| - 2|N_1|(\Omega + \langle M_2 \rangle)^{-1} \langle A \rangle - \langle A \rangle - \langle M_1 \rangle \right) \\
&= I - 2(\Omega + \langle M_1 \rangle)^{-1} \left( |N_1|(\Omega + \langle M_2 \rangle)^{-1} + I \right) \langle A \rangle. \tag{4.13}
\end{align*}
\]

Note that, \( \Omega + \langle M_2 \rangle \) is an \( M \)-matrix therefore \( (\Omega + \langle M_2 \rangle)^{-1} \geq 0 \) and clearly \( |N_1|(\Omega + \langle M_2 \rangle)^{-1} + I > 0 \). Since \( A \) is an \( H_+ \)-matrix, which means \( \langle A \rangle \) is an \( M \)-matrix. Based on Lemma 2.3, there exists a positive vector \( u > 0 \) such that \( \langle A \rangle u > 0 \). Notice that the positive vector \( u \) is only dependent on the matrix \( A \) but not on the splittings \( A = M_1 - N_1 = M_2 - N_2 \). Thus

\[
Su = u - 2(\Omega + \langle M_1 \rangle)^{-1} \left( |N_1|(\Omega + \langle M_2 \rangle)^{-1} + I \right) \langle A \rangle u < u,
\]

based on Lemma 2.5, conclude that \( \rho(S) < 1 \). This completes the proof. \( \square \)

**Theorem 4.2.** Let \( \Omega \in \mathbb{R}^{n \times n} \) be a positive diagonal matrix, \( A = M_1 - N_1 = M_2 - N_2 \) be two splittings of the positive-definite matrix \( A \in \mathbb{R}^{n \times n} \) with \( M_1, M_2 \in \mathbb{R}^{n \times n} \) being positive-definite. Define

\[
\xi = \| (\Omega + M_1)^{-1} (|N_1|((\Omega + M_2)^{-1})|N_2| + |N_1|)|((\Omega + M_2)^{-1})|\Omega - A| + |\Omega - A| \|,
\]

where \( \| \cdot \| \) is an arbitrary matrix norm. If \( \xi < 1 \), then for any two initial vectors \( x^{(0)}, x^{(1)} \in \mathbb{R}^n \), the iteration sequence \( \{ x^{(k)} \}_{k=0}^{+\infty} \) generated by Method 3.3 converges to the exact solution of \( \text{LCP}(q, A) \).

**Proof.** It follows from (4.11)

\[
\begin{vmatrix}
|x^{(k+1)} - x^*| \\
|x^k| - |x^*|
\end{vmatrix} \leq RX,
\]

where

\[
R = \begin{bmatrix}
v_1 & u_2 \\
I & 0
\end{bmatrix} \geq 0,
\]

\[
v_1 = |(\Omega + M_1)^{-1}|\Omega - A| + |(\Omega + M_1)^{-1}|N_1|\Omega - A|,
\]

\[
v_2 = |(\Omega + M_1)^{-1}|N_1|\Omega - A|,
\]

and

\[
X = \begin{vmatrix}
|x^{(k)} - x^*| \\
|x^{(k-1)} - x^*| - |x^*|
\end{vmatrix} \leq \begin{vmatrix}
|x^{(k)} - x^*| \\
|x^{(k-1)} - x^*|
\end{vmatrix}.
\]

If \( \rho(R) < 1 \), then for any two initial vectors, the iteration sequence \( \{ x^{(k)} \}_{k=0}^{+\infty} \subset \mathbb{R}^n \) generated by TSTM converges and implies convergence of the sequence \( \{ z^{(k)} \}_{k=0}^{+\infty} \) to
the unique solution \( z^* \in R^n_+ \) of the LCP\((q, A)\). According to Lemma 2.1, we only need to show that

\[
\rho \left( |(\Omega + M_1)^{-1}| |\Omega - A| + |(\Omega + M_1)^{-1}| |N_1| |(\Omega + M_2)^{-1}| |N_2| + |(\Omega + M_1)^{-1}| |N_1| |(\Omega + M_2)^{-1}| |\Omega - A| \right) < 1,
\]

or

\[
\rho \left( |(\Omega + M_1)^{-1}| |N_1| |(\Omega + M_2)^{-1}| |N_2| + |N_1| |(\Omega + M_2)^{-1}| |\Omega - A| + |\Omega - A| \right) < 1.
\]

As

\[
\rho \left( |(\Omega + M_1)^{-1}| |N_1| |(\Omega + M_2)^{-1}| |N_2| + |N_1| |(\Omega + M_2)^{-1}| |\Omega - A| + |\Omega - A| \right) \leq \| |(\Omega + M_1)^{-1}| |N_1| |(\Omega + M_2)^{-1}| |N_2| + |N_1| |(\Omega + M_2)^{-1}| |\Omega - A| + |\Omega - A| \| =: \xi,
\]

if \( \xi < 1 \) then the iteration sequence \( \{z^{(k)}\}_{k=0}^{+\infty} \subset R^n \) generated by TSTM method converges to the unique solution \( z^* \in R^n_+ \) of the \( \text{LCP}(q, A) \) for any two initial vectors.

\[ \square \]

**Lemma 4.1.** Let \( A \) is an \( H_{+} \)-matrix with \( B \neq 0 \). If

\[
0 < \alpha < \frac{1}{\rho(D^{-1}|B|)},
\]

there exists a positive vector \( v \in R^n \), such that

\[
((1 + \alpha - |1 - \alpha|)D - 2\alpha|B|) v > 0.
\]

**Proof.** As \( A \) is an \( H_{+} \)-matrix, we have \( \rho(D^{-1}|B|) < 1 \), see [5]. By direct computation,

\[
0 < \alpha < \frac{1}{\rho(D^{-1}|B|)}
\]

implies

\[
(1 + \alpha - |1 - \alpha| - 2\alpha\rho(D^{-1}|B|) > 0.
\]

Based on proof of [51, Theorem 4.2] there exists a positive vector \( v \), such that

\[
((1 + \alpha - |1 - \alpha|)D - 2\alpha|B|) v > 0.
\]

The proof is complete. \[ \square \]

**Case I.** If \( 0 < \alpha \leq 1 \), then we have that

\[
(1 + \alpha - |1 - \alpha|)D - 2\alpha|B| = 2\alpha\langle A \rangle
\]

is an \( M \)-matrix.
Case II. If $1 < \alpha < \frac{1}{\rho(D^{-1}|B|)}$, then we have

$$(1 + \alpha - |1 - \alpha|)D - 2\alpha|B| = 2D(I - \alpha D^{-1}|B|).$$

As $I - \alpha D^{-1}|B|$ is an $M$-matrix thus $2D(I - \alpha D^{-1}|B|)$ is an $M$-matrix. Accordingly there exists a positive vector $v$, such that $((1 + \alpha - |1 - \alpha|)D - 2\alpha|B|)v > 0$.

**Theorem 4.3.** Let $A \in \mathbb{R}^{n \times n}$ be an $H_+$-matrix, $\Omega \in \mathbb{R}^{n \times n}$ be a positive diagonal matrix, $A = M_1 - N_1 = M_2 - N_2$ be two $H$- compatible splitting of $A$, be given by (3.6). Assume that $\Omega \geq D$, where $D = \text{diag}(A)$. Then the iteration sequence $\{z^{(k)}\}_{k=0}^{+\infty}$ generated by Method 3.3 converges to the unique solution $z^* \in \mathbb{R}^n$ of the LCP($q, A$) for any two initial vectors $x^{(0)}$ and $x^{(1)}$, provided

$$0 < \beta \leq \alpha < \frac{1}{\rho(D^{-1}|B|)}.$$  

**Proof.** From the proof of Theorem 4.1, the Method 3.3 converges if $\rho(S) < 1$, where

$$S = (\Omega + \langle A \rangle)^{-1}|\Omega - A| + (\Omega + \langle M_1 \rangle)^{-1}|N_1(\Omega + \langle M_2 \rangle)^{-1}|N_2|
+ (\Omega + \langle M_1 \rangle)^{-1}|N_1(\Omega + \langle M_2 \rangle)^{-1}|\Omega - A|
= (\Omega + \langle M_1 \rangle)^{-1}\left(\langle N_1(\Omega + \langle M_2 \rangle)^{-1}|N_2| + \Omega - \langle A \rangle \right) + \Omega - \langle A \rangle). \quad (4.15)$$

Based on Lemma 4.1, there exists a positive vector $v$, such that $((1 + \alpha - |1 - \alpha|)D - 2\alpha|B|)v > 0$. Therefore,

$$(\Omega + \langle M_2 \rangle)^{-1}|N_2| + \Omega - \langle A \rangle)v
\leq (\alpha\Omega + D - \beta|U|)^{-1}(1 - \alpha|D + |\alpha - \beta||U| + \alpha|L| + \alpha\Omega - \alpha D + \alpha|B|)v
= \left(I - (\alpha\Omega + D - \beta|U|)^{-1}(1 + \alpha - |1 - \alpha|)D - 2\alpha|B|\right)v
= v - (\alpha\Omega + D - \beta|U|)^{-1}\left((1 + \alpha - |1 - \alpha|)D - 2\alpha|B|\right)v < v. \quad (4.16)$$

From (4.15) and (4.16)

$$Sv = (\Omega + \langle M_1 \rangle)^{-1}\left(\langle |N_1(\Omega + \langle M_2 \rangle)^{-1}|N_2| + \Omega - \langle A \rangle \right) + \Omega - \langle A \rangle)v
\leq (\Omega + \langle M_1 \rangle)^{-1}\left(|N_1|\Omega + \langle M_2 \rangle)^{-1}|N_2| + \Omega - \langle A \rangle\right)v + \Omega v - \langle A \rangle)v
\leq (\alpha\Omega + D - \beta|L|)^{-1}(1 - \alpha|D + |\alpha - \beta||L| + \alpha|U| + \alpha\Omega - \alpha D + \alpha|B|)\v
= \left(I - (\alpha\Omega + D - \beta|L|)^{-1}(1 + \alpha - |1 - \alpha|)D - 2\alpha|B|\right)v
= v - (\alpha\Omega + D - \beta|L|)^{-1}\left((1 + \alpha - |1 - \alpha|)D - 2\alpha|B|\right)v < v$$
based on Lemma 2.5, conclude that \( \rho(S) < 1 \). This completes the proof. Note that, we have used the facts that \( \Omega \geq D \) and \( |L| + |U| = |B| \).

Now we prove the convergence of Method 3.4 when the system matrix is either a positive-definite or an \( H_+ \)-matrix.

**Theorem 4.4.** Let \( \Omega \in \mathbb{R}^{n \times n} \) be a positive diagonal matrix, \( A = M_1 - N_1 = M_2 - N_2 \) be two splittings of the positive-definite matrix \( A \in \mathbb{R}^{n \times n} \), with \( M_1, M_2 \in \mathbb{R}^{n \times n} \) being positive-definite. Define

\[
\tau = \left\| (\Omega + M_1)^{-1}|(N_1| + |\Omega - A)| \right\| \left\| (\Omega + M_2)^{-1}|(N_2| + |\Omega - A)| \right\|
\]

where \( \| \cdot \| \) is an arbitrary matrix norm. If \( \tau < 1 \), then for any two initial vectors \( x^{(0)} \), \( x^{(1)} \in \mathbb{R}^n \), the iteration sequence \( \{z^{(k)}\}^\infty_{k=0} \) generated by Method 3.4 converges to the exact solution of \( \text{LCP}(q, A) \).

**Proof.** Let \( z^* \) be a solution of \( \text{LCP}(q, A) \), then

\[
x^* = \left( \frac{\gamma}{2} \right) (z^* - \Omega^{-1}w^*)
\]

with

\[
|\ x^* | = \left( \frac{\gamma}{2} \right) (z^* + \Omega^{-1}w^*)
\]

satisfies the implicit fixed-point equations

\[
\begin{cases}
(\Omega + M_2)x^* = N_2x^* + (\Omega - A)|x^*| - \gamma q, \\
(\Omega + M_1)x^* = N_1x^* + (\Omega - A)|x^*| - \gamma q.
\end{cases}
\]

(4.17)

From (3.5) and (4.17), we obtain

\[
\begin{cases}
x^{(k + \frac{1}{2})} = (\Omega + M_2)^{-1}N_2x^{(k)} + (\Omega + M_2)^{-1}(\Omega - A)|x^{(k-1)}| \\
& - (\Omega + M_2)^{-1}\gamma q, \\
x^{(k+1)} = (\Omega + M_1)^{-1}N_1x^{(k + \frac{1}{2})} + (\Omega + M_1)^{-1}(\Omega - A)|x^{(k+\frac{1}{2})}| \\
& - (\Omega + M_1)^{-1}\gamma q,
\end{cases}
\]

(4.18)

and

\[
\begin{cases}
x^* = (\Omega + M_2)^{-1}N_2x^* + (\Omega + M_2)^{-1}(\Omega - A)|x^*| - (\Omega + M_2)^{-1}\gamma q, \\
x^* = (\Omega + M_1)^{-1}N_1x^* + (\Omega + M_1)^{-1}(\Omega - A)|x^*| - (\Omega + M_1)^{-1}\gamma q.
\end{cases}
\]

(4.19)

By subtracting (4.19) from (4.18), we get

\[
\begin{cases}
x^{(k + \frac{1}{2})} - x^* = (\Omega + M_2)^{-1}N_2(x^{(k)} - x^*) + (\Omega + M_2)^{-1}(\Omega - A)((x^{(k-1)}| - |x^*)|, \\
x^{(k+1)} - x^* = (\Omega + M_1)^{-1}N_1(x^{(k + \frac{1}{2})} - x^*) + (\Omega + M_1)^{-1}(\Omega - A)((x^{(k + \frac{1}{2})}| - |x^*)|.
\end{cases}
\]
Hence, by using the triangle inequality \( ||x^{(k)}| - |x^*|| \leq |x^{(k)} - x^*| \) we have
\[
|x^{(k+\frac{1}{2})} - x^*| = |(\Omega + M_2)^{-1}N_2(x^{(k)} - x^*) + (\Omega + M_2)^{-1}(\Omega - A)(|x^{(k-1)}| - |x^*|)|
\leq |(\Omega + M_2)^{-1}N_2||x^{(k)} - x^*| + |(\Omega + M_2)^{-1}(\Omega - A)||x^{(k-1)} - x^*|
\leq |(\Omega + M_2)^{-1}||N_2||x^{(k)} - x^*| + |\Omega - A||x^{(k-1)} - x^*|.
\]

(4.20)

Similarly,
\[
|x^{(k+1)} - x^*| = |(\Omega + M_1)^{-1}N_1(x^{(k+\frac{1}{2})} - x^*) + (\Omega + M_1)^{-1}(\Omega - A)(|x^{(k+\frac{1}{2})}| - |x^*|)|
\leq |(\Omega + M_1)^{-1}N_1||x^{(k+\frac{1}{2})} - x^*| + |(\Omega + M_1)^{-1}(\Omega - A)||x^{(k+\frac{1}{2})} - x^*|
\leq |(\Omega + M_1)^{-1}||N_1||x^{(k)} - x^*| + |\Omega - A||x^{(k-1)} - x^*|.
\]

(4.21)

We consider
\[
\psi_1 = |(\Omega + M_1)^{-1}||N_1| + |\Omega - A|,
\]

therefore by (4.20) and (4.21), we have
\[
|x^{(k+1)} - x^*| \leq \psi_1|x^{(k+\frac{1}{2})} - x^*|
\leq \psi_1|(\Omega + M_2)^{-1}||N_2||x^{(k)} - x^*| + |\Omega - A||x^{(k-1)} - x^*|).
\]

By adding the identical relation \( |x^{(k)} - x^*| = |x^{(k)} - x^*| \) we obtain,
\[
\left\{ \begin{array}{l}
|x^{(k+1)} - x^*| \leq \psi_1|(\Omega + M_2)^{-1}||N_2||x^{(k)} - x^*| + |\Omega - A||x^{(k-1)} - x^*|, \\
|x^{(k)} - x^*| = |x^{(k)} - x^*|.
\end{array} \right.
\]

(4.22)

In other words
\[
|\begin{bmatrix} x^{(k+1)} - x^* \\ x^{(k)} - x^* \end{bmatrix} | \leq \begin{bmatrix} \psi_1|(\Omega + M_2)^{-1}||N_2| & \psi_1|(\Omega + M_2)^{-1}||\Omega - A| \\ I & 0 \end{bmatrix} \begin{bmatrix} x^{(k)} - x^* \\ x^{(k-1)} - x^* \end{bmatrix}.
\]

(4.23)

Assume that
\[
W = \begin{bmatrix} \psi_1|(\Omega + M_2)^{-1}||N_2| & \psi_1|(\Omega + M_2)^{-1}||\Omega - A| \\ I & 0 \end{bmatrix},
\]

(4.24)

if \( \rho(W) < 1 \), then for any two initial vectors, the iteration sequence \( \{x^{(k)}\}_{k=0}^{+\infty} \subset \mathbb{R}^n \) generated by Method 3.4 converges and implies convergence of the sequence \( \{z^{(k)}\}_{k=0}^{+\infty} \) to the unique solution \( z^* \in \mathbb{R}^n \) of the LCP\((q, A)\). Since
\[
\psi_1 = |(\Omega + M_1)^{-1}||N_1| + |\Omega - A|
\]
is a nonnegative matrix, obviously, $W \geq 0$. According to Lemma 2.1, we only need to show that
\[ \rho \left( \psi_1 | (\Omega + M_2)^{-1} | N_2 | + \psi_1 | (\Omega + M_2)^{-1} | \Omega - A \right) < 1, \]
or equivalently
\[ \rho \left( \psi_1 | (\Omega + M_2)^{-1} | (N_2 + | \Omega - A |) \right) =: \rho(\psi_1 \psi_2) < 1, \quad (4.25) \]
where
\[ \psi_2 = | (\Omega + M_2)^{-1} | (N_2 + | \Omega - A |). \]
Regarding (4.25), to prove the convergence, it is sufficient to show that $\rho(\psi_1 \psi_2) < 1$. We have
\[ \rho(\psi_1 \psi_2) \leq \| \psi_1 \psi_2 \| \leq \| \psi_1 \| \| \psi_2 \| =: \tau, \]
where
\[ \tau = \left\| \left( | (\Omega + M_1)^{-1} | (N_1 + | \Omega - A |) \right) \left( | (\Omega + M_2)^{-1} | (N_2 + | \Omega - A |) \right) \right\|. \]
If $\tau < 1$, then the iteration sequence $\{ z^{(k)} \}_{k=0}^{+\infty} \subset R^n$ generated by Method 3.4 converges to the unique solution $z^* \in R^n_+$ of the LCP($q, A$) for any two initial vectors.

**Theorem 4.5.** Let $A \in R^{n \times n}$ be an $H_+$-matrix, $A = M_1 - N_1 = M_2 - N_2$ be two $H$-splittings of $A$, $\Omega \in R^{n \times n}$ be a positive diagonal matrix. Assume that there is a positive diagonal matrix $G \in R^{n \times n}$ such that $\langle A \rangle G$, $(\langle M_1 \rangle - | N_1 \rangle) G$, and $(\langle M_2 \rangle - | N_2 \rangle) G$ are s.d.d. matrices. If one of the following inequalities holds:

(I) $\Omega \geq D$, where $D = \text{diag}(A),$

(II) $\frac{1}{2} | \langle M_i \rangle - | N_i \rangle | Ge \leq \Omega Ge \leq DGe$, where $e = (1, 1, \ldots, 1)^T \in R^n$, $D = \text{diag}(A)$ and $i = 1, 2.$

Then for any two initial vectors $x^{(0)}$ and $x^{(1)}$ the iteration sequence $\{ z^{(k)} \}_{k=0}^{+\infty}$ generated by Method 3.4 converges to the unique solution $z^*$ of the LCP($q, A$).

**Proof.** To prove the convergence of Method 3.4. Based on (4.23) and (4.24), we only need to demonstrate $\rho(W) < 1$. As $W \geq 0$, according to Lemma 2.1 and (4.25), it is enough to show $\rho(\psi_1 \psi_2) < 1$ where
\[ \psi_i = \left| (\Omega + M_i)^{-1} \right| (| N_i | + | \Omega - A |), \quad i = 1, 2. \]
Since $(\langle M_i \rangle - | N_i \rangle) G$ are s.d.d. matrices, we obtain
\[ 0 < (\langle M_i \rangle - | N_i \rangle) Ge \leq \langle M_i \rangle Ge \leq (\Omega + \langle M_i \rangle) Ge, \quad i = 1, 2, \quad (4.26) \]
which implies that $(\Omega + \langle M_i \rangle) G$ are s.d.d. matrices and $\Omega + M_i$ are $H_+$-matrices. According to Lemma 2.2, we have
\[ 0 \leq \left| (\Omega + M_i)^{-1} \right| \leq (\Omega + \langle M_i \rangle)^{-1}, \]
Let \( > \) From (4.26) and according to the assumption of this theorem we have

\[
\psi_i = \frac{1}{2} \left( (\Omega + \langle M_i \rangle)^{-1} ([N_i] + |\Omega - A|) \right)
\leq \left( \Omega + \langle M_i \rangle \right)^{-1} ([N_i] + |\Omega - A|) =: \varphi_i, \quad i = 1, 2,
\]

where

\[
\varphi_i = \left( \Omega + \langle M_i \rangle \right)^{-1} ([N_i] + |\Omega - A|), \quad i = 1, 2.
\]

Then \( 0 \leq \psi_1 \psi_2 \leq \varphi_1 \varphi_2 \) which implies \( \rho(\psi_1 \psi_2) \leq \rho(\varphi_1 \varphi_2) \). In other words, we only need to prove \( \rho(\varphi_1 \varphi_2) < 1 \).

**Case I.** If \( \Omega \geq D \), we have

\[
\varphi_i = \left( \Omega + \langle M_i \rangle \right)^{-1} ([N_i] + |\Omega - A|)
= \left( \Omega + \langle M_i \rangle \right)^{-1} ([N_i] + |\Omega - D| + |B|)
= \left( \Omega + \langle M_i \rangle \right)^{-1} ([N_i] - \langle M_i \rangle + \Omega - \langle A \rangle)
= I - \left( \Omega + \langle M_i \rangle \right)^{-1} (\langle M_i \rangle - [N_i] + \langle A \rangle), \quad i = 1, 2.
\]

From (4.26) and according to the assumption of this theorem we have \( (\langle M_2 \rangle - [N_2] + \langle A \rangle) Ge > 0 \) and \( (\langle A \rangle) Ge > 0 \). Thus

\[
(\langle M_i \rangle - [N_i] + \langle A \rangle) Ge > 0, \quad i = 1, 2.
\]

Let \( Ge = v \in R^n_+ \), by combining \( (\langle M_i \rangle - [N_i] + \langle A \rangle)v > 0 \) and \( (\Omega + \langle M_i \rangle)^{-1} \) is a nonnegative matrix without zero rows, it follows that

\[
\varphi_1 \varphi_2 v = \varphi_1 \left( I - \left( \Omega + \langle M_2 \rangle \right)^{-1} (\langle M_2 \rangle - [N_2] + \langle A \rangle) \right) v
= \varphi_1 \left( v - \left( \Omega + \langle M_2 \rangle \right)^{-1} (\langle M_2 \rangle - [N_2] + \langle A \rangle) v \right) < \varphi_1 v,
\]

also

\[
\varphi_1 v = \left( I - \left( \Omega + \langle M_1 \rangle \right)^{-1} (\langle M_1 \rangle - [N_1] + \langle A \rangle) \right) v < v,
\]

thus \( \varphi_1 \varphi_2 v < v \). Based on Lemma 2.5, we conclude that \( \rho(\varphi_1 \varphi_2) < 1 \), therefore \( \rho(\psi_1 \psi_2) < 1 \), which completes the proof in Case I.

**Case II.** If \( \frac{1}{2} ([A] - \langle M_i \rangle + [N_i]) Ge < \Omega Ge \leq D Ge \), then the next inequality holds

\[
\varphi_i = \left( \Omega + \langle M_i \rangle \right)^{-1} ([N_i] + |\Omega - A|)
= \left( \Omega + \langle M_i \rangle \right)^{-1} ([N_i] + \langle D - \Omega \rangle + |B|)
= \left( \Omega + \langle M_i \rangle \right)^{-1} ([N_i] - \langle M_i \rangle + \langle M_i \rangle + \Omega - 2\Omega + |A|)
= I - \left( \Omega + \langle M_i \rangle \right)^{-1} (2\Omega - |A| + \langle M_i \rangle - [N_i]), \quad i = 1, 2.
\]

As \( \frac{1}{2} ([A] - \langle M_i \rangle + [N_i]) Ge < \Omega Ge \leq D Ge \), and \( (\Omega + \langle M_i \rangle)^{-1} \) is a nonnegative matrix without zero rows, then \( (\Omega + \langle M_i \rangle)^{-1} (2\Omega - |A| + \langle M_i \rangle - [N_i]) Ge > 0 \). Let \( Ge = v \in R^n_+ \), we have \( (\Omega + \langle M_i \rangle)^{-1} (2\Omega - |A| + \langle M_i \rangle - [N_i]) v > 0 \). Accordingly

\[
\varphi_1 \varphi_2 v = \varphi_1 \left( \Omega + \langle M_2 \rangle \right)^{-1} ([N_2] + |\Omega - A|) v.
\]
\[
\varphi_1 \left( I - (\Omega + \langle M_2 \rangle)^{-1} (2\Omega - |A| + \langle M_2 \rangle - |N_2|) \right) v < \varphi_1 v.
\]

On the other hand
\[
\varphi_1 v = \left( I - (\Omega + \langle M_1 \rangle)^{-1} (2\Omega - |A| + \langle M_1 \rangle - |N_1|) \right) v < v,
\]

therefore, \( \varphi_1 \varphi_2 v < v \). From Lemma 2.5, we conclude that \( \rho(\varphi_1 \varphi_2) < 1 \), thus \( \rho(\psi_1 \psi_2) < 1 \). This completes the proof. \( \quare \)

**Remark 4.1.**

- For some special cases, we can compute the positive diagonal matrix \( G \) given in the assumption (II), see [55].

- In Theorem 4.5, if \( A = M_1 - N_1 = M_2 - N_2 \) are two \( H \)-compatible splittings of \( A \) (\( H \)-splittings), then based on Lemma 2.3 there exists a positive diagonal matrix \( G \), such that \( \langle A \rangle G = \langle \langle M_1 \rangle - |N_1| \rangle G = \langle \langle M_2 \rangle - |N_2| \rangle G \) are s.d.d. matrices with positive diagonal entries.

**Theorem 4.6.** Assume that \( A \in \mathbb{R}^{n \times n} \) is an \( H_+ \)-matrix, and \( A = M_1 - N_1 = M_2 - N_2 \) is given by (3.6) and \( \Omega \in \mathbb{R}^{n \times n} \) is a positive diagonal matrix. Then TM II AOR method converges for any initial vectors \( x^{(0)}, x^{(1)} \in \mathbb{R}^n \), when

\[
0 < \beta \leq \alpha < \frac{1}{\rho(D^{-1}|B|)}.
\]

**Proof.** In Theorem 4.5, we illustrated that to prove convergence, it suffices that only \( \rho(\varphi_1 \varphi_2) < 1 \) where

\[
\varphi_i = (\Omega + \langle M_i \rangle)^{-1} (|N_i| + |\Omega - A|), \quad i = 1, 2.
\]

Based on [54, Theorem 3.2], we obtain \( \rho(\varphi_1 \varphi_2) < 1 \). Which implies the TM II MAOR method is convergent. \( \square \)

**5. Numerical experiments**

In this section, with three examples, we illustrate the effectiveness of the proposed methods in terms of iteration steps (IT), CPU time in seconds (CPU), and norm of absolute residual (RES).

All the computations were run on an Intel(R) Core(TM), where the CPU is 2.10 GHz, the memory is 8:00 GB, and the programming language is Matlab R2016a. All iterations are terminated either \( RES(z^{(k)}) \leq 10^{-5} \) or the maximum iteration numbers exceed 10000. In our numerical computations, we choose, \( \Omega = \frac{1}{2n} D \). All initial vectors in our tests are the same

\[
x^{(0)} = x^{(1)} = (1, 0, 1, 0, \ldots, 1, 0, \ldots)^T \in \mathbb{R}^n.
\]
Two-Step Two-Sweep Modulus-Based Matrix Splitting Iteration Method for LCP

Here

\[ \text{RES}(z^{(k)}) := \| \min (Az^{(k)} + q, z^{(k)}) \|_2. \]

The results are briefly summarized in Tables 1-4, to compare the performance of the proposed methods (TM II SOR, TSMSOR) with the methods TSMSOR, TMSOR, MSOR.

**Example 5.1** ([2]). Let \( m \) be a prescribed positive integer and \( n = m^2 \). Consider the LCP\((q, A)\), for which \( A \in R^{n \times n} \) is given by \( A = A + \mu I \) and vector \( q = -Az^* \in R^n \), where

\[
\hat{A} = \begin{bmatrix}
S & -I & & \\
-I & S & -I & \\
& -I & S & \\
& & & \ddots \\
& & & & -I \\
& & & & -I \\
& & & & -I \\
\end{bmatrix} \in R^{n \times n}
\]

is a block-tridiagonal matrix and \( S = \text{tridiag}(-1, 4, -1) \in R^{m \times m} \) is a tridiagonal matrix, and \( z^* = (1, 2, 1, 2, \ldots) \in R^n \) is the unique solution of the LCP\((q, A)\).

**Example 5.2** ([2]). Let \( m \) be a prescribed positive integer and \( n = m^2 \). Consider the LCP\((q, A)\), for which \( A \in R^{n \times n} \) is given by \( A = A + \mu I \) and vector \( q = -Az^* \in R^n \), where

\[
\hat{A} = \begin{bmatrix}
S & -1.5I & & \\
-0.5I & S & -1.5I & \\
& -0.5I & S & \\
& & & \ddots \\
& & & & -1.5I \\
& & & & -0.5I \\
& & & & 0.5I \\
\end{bmatrix} \in R^{n \times n}
\]

is a block-tridiagonal matrix, and \( S = \text{tridiag}(-0.5, 4, -1.5) \in R^{m \times m} \) is a tridiagonal matrix, \( z^* = (1, 2, 1, 2, \ldots)^T \in R^n \) is the unique solution, and \( I \in R^{n \times n} \) is the identity matrix.

**Remark 5.1.** Note that in Example 5.1, the system matrix \( A \in R^{n \times n} \) is symmetric positive-definite for \( \mu \geq 0 \) and in Example 5.2, the system matrix \( A \in R^{n \times n} \) is nonsymmetric and weakly diagonal dominant for \( \mu \geq 0 \). The LCP\((q, A)\) has a unique solution for both examples. In Tables 1-4, the numerical results for Examples 5.1-5.3 are reported. It should be mentioned that the selected iteration parameter \( \alpha \) for Methods 3.3 and 3.4 must satisfy in Theorems 4.3 and 4.6, respectively. When \( \alpha = 1 \), two-step two-sweep modulus-based successive overrelaxation (TSTMSOR) method is reduced to two-step two-sweep modulus-based Gauss-Seidel (TSTMGS) method, two-sweep
Table 1: Numerical results for Examples 5.1 and 5.2 with $m = 70$ and $\alpha = 1$.

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>Method</th>
<th>Example 5.1</th>
<th>Example 5.2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>CPU</td>
<td>IT</td>
</tr>
<tr>
<td>0.2</td>
<td>TM II SOR</td>
<td>2.9794</td>
<td>92</td>
</tr>
<tr>
<td></td>
<td>TSTMSOR</td>
<td>3.3767</td>
<td>104</td>
</tr>
<tr>
<td></td>
<td>MSOR</td>
<td>5.2853</td>
<td>326</td>
</tr>
<tr>
<td>0.5</td>
<td>TM II SOR</td>
<td>1.3093</td>
<td>41</td>
</tr>
<tr>
<td></td>
<td>TSTMSOR</td>
<td>1.5416</td>
<td>47</td>
</tr>
<tr>
<td></td>
<td>TMSOR</td>
<td>3.0459</td>
<td>96</td>
</tr>
<tr>
<td></td>
<td>MSOR</td>
<td>3.4231</td>
<td>212</td>
</tr>
<tr>
<td>1</td>
<td>TM II SOR</td>
<td>0.73978</td>
<td>23</td>
</tr>
<tr>
<td></td>
<td>TSTMSOR</td>
<td>0.94442</td>
<td>29</td>
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<tr>
<td></td>
<td>TMSOR</td>
<td>1.6888</td>
<td>53</td>
</tr>
<tr>
<td></td>
<td>MSOR</td>
<td>1.2881</td>
<td>80</td>
</tr>
<tr>
<td>1.5</td>
<td>TM II SOR</td>
<td>0.65855</td>
<td>21</td>
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<tr>
<td></td>
<td>TSTMSOR</td>
<td>0.96567</td>
<td>30</td>
</tr>
<tr>
<td></td>
<td>TMSOR</td>
<td>1.2152</td>
<td>38</td>
</tr>
<tr>
<td></td>
<td>MSOR</td>
<td>1.1456</td>
<td>71</td>
</tr>
</tbody>
</table>

Numerical results reported in Table 1 indicate that, for Examples 5.1 and 5.2 with $m = 70$ and $\alpha = 1$, fixing the value of $\mu$, the proposed methods (TSTMSOR, TM II SOR) have better performance than other three mentioned methods, in terms of CPU time and iteration steps. Notice that the TM II SOR method is more efficient than the TSTMSOR method. In addition, as $\mu$ increases, CPU time and iteration steps decrease. In Table 2, with $\alpha = 1$, for different problem sizes of $m$, fixing the value of $\mu$, we observe that by increasing the problem size $m$, the iteration steps and CPU time of TM II SOR, TSTMSOR, TMSOR, TMSOR, and MSOR methods ascend. Indeed, as the problem size increases, the proposed methods are still superior to other methods in terms of iteration steps and CPU time.
Table 2: Numerical results for Examples 5.1 and 5.2 with $\alpha = 1$.

<table>
<thead>
<tr>
<th>Example</th>
<th>Method</th>
<th>$\mu$</th>
<th>$m=50$</th>
<th>CPU</th>
<th>IT</th>
<th>RES</th>
<th>$m=70$</th>
<th>CPU</th>
<th>IT</th>
<th>RES</th>
<th>$m=90$</th>
<th>CPU</th>
<th>IT</th>
<th>RES</th>
</tr>
</thead>
<tbody>
<tr>
<td>Example 5.1</td>
<td>TM II SOR</td>
<td>$\mu = 0.1$</td>
<td>1.3128</td>
<td>160</td>
<td>9.6332e-06</td>
<td>5.2588</td>
<td>170</td>
<td>9.3316e-06</td>
<td>15.0876</td>
<td>175</td>
<td>9.878e-06</td>
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<tr>
<td></td>
<td>TM SOR</td>
<td></td>
<td>3.16</td>
<td>384</td>
<td>9.8187e-06</td>
<td>12.5838</td>
<td>405</td>
<td>9.7736e-06</td>
<td>35.3347</td>
<td>413</td>
<td>9.9183e-06</td>
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<td>MSOR</td>
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<td>1.527</td>
<td>360</td>
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<td>5.4965</td>
<td>382</td>
<td>9.7892e-06</td>
<td>17.0462</td>
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<td>430</td>
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<td>7.366e-06</td>
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<td>82</td>
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<td>MSOR</td>
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<td>1.2203</td>
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<td>TM II SOR</td>
<td>$\mu = 1.2$</td>
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<td></td>
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<td>6.3841e-06</td>
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<td>Example 5.2</td>
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<td>9.0389e-06</td>
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<td>22</td>
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<td>9.2696e-06</td>
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<td>9.8995e-06</td>
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</table>

In Tables 1 and 2, we examined the cases in which $\alpha = 1$. As mentioned earlier, the TM II SOR method and the TSMSOR method are convergent when $\alpha$ satisfies in Theorems 4.6 and 4.3, respectively. Table 3 reports the numerical results for the values of $\alpha \neq 1$, which satisfies Theorems 4.6 and 4.3. The results indicate that for $\alpha > 1$, the new methods are superior in terms of running time and of iteration steps (for Examples 5.1 and 5.2). In addition, it is observed that for some $\alpha$ values the compared methods are divergent while the proposed methods is convergent. In fact, according to Table 3, it can be concluded that the TM II SOR and the TSMSOR method have wider convergence regions than the other three mentioned methods (TSMSOR, TM-SOR, MSOR). Besides, the results show the superiority of the proposed methods by increasing the size of the problem is preserved. Further, we can see that the TM II SOR method is more efficient than the TSMSOR method.

In the following we have an application of proposed methods for solving the linear complementarity problems from Black-Scholes American option pricing [35].
<table>
<thead>
<tr>
<th>Example and m</th>
<th>Method</th>
<th>α</th>
<th>0.6</th>
<th>0.9</th>
<th>1.1</th>
<th>1.2</th>
<th>1.25</th>
<th>1.3</th>
</tr>
</thead>
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<td>CPU</td>
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<td>0.058791</td>
<td>0.068433</td>
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<td>0.066359</td>
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<tr>
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<td>20</td>
<td>20</td>
<td>19</td>
<td>19</td>
</tr>
<tr>
<td></td>
<td></td>
<td>RES</td>
<td>6.2102e-06</td>
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<td>4.804e-06</td>
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<td>TSTMSOR</td>
<td>CPU</td>
<td>0.06876</td>
<td>0.070093</td>
<td>0.097064</td>
<td>0.14077</td>
<td>0.16332</td>
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<td>28</td>
<td>40</td>
<td>46</td>
<td>51</td>
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Table 3: Numerical results for Examples 5.1 and 5.2 with $\mu = 4$ and $\alpha \neq 1$. 
Table 3: Numerical results for Examples 5.1 and 5.2 with $\mu = 4$ and $\alpha \neq 1.$ (cont’d)

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<td>8.4904e-06</td>
<td>569.4081</td>
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</table>
Example 5.3. Based on Black–Scholes model, the price \( u(x, t) \) of American put options satisfies the partial differential complementarity condition

\[
\left( \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} \right) (u(x, t) - g(x, t)) = 0, \\
\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} \geq 0, \quad u(x, t) - g(x, t) \geq 0
\]

with

\[
u(x, 0) = g(x, 0), \quad \lim_{x \to \pm \infty} u(x, t) = \lim_{x \to \pm \infty} g(x, t), \quad (x, t) \in (-\infty, +\infty) \times [0, T].
\]

To discretize (5.1), we need to limit \( x \in [a, b] \) and choose the values of \( a \) and \( b \) based on the approach in [35]. Using the forward difference scheme for time \( t \) and implicit difference scheme for the price \( x \) leads to the following linear complementarity problem:

\[
Az - q \geq 0, \quad z - g \geq 0, \quad (Az - q)(z - g) = 0
\]

with \( A = \text{tridiag}(-\tau, 1 + 2\tau, -\tau) \) and \( \tau = \frac{\Delta t}{(\Delta x)^2} \), \( \Delta t \) denotes the time step and \( \Delta x \) denotes the price step. The vector \( q \) is to be adjusted such that \( q = Ae, \quad e = (1, 1, \ldots, 1)^T \).

Table 4 reports iteration steps, the CPU times and the residual norms of TM II SOR, TSTMSOR, TSMSOR, TMSOR, and MSOR methods (Note that using the aforementioned initial vectors and termination condition). This table is adjusted for different problem size of \( m \) when \( \tau = 3 \).

6. Conclusions

This paper presented two effective new methods for solving \( \text{LCP}(q, A) \). The convergence analysis of these methods was investigated under suitable conditions. The comparison of the numerical results obtained from solving Examples 5.1-5.3 through five methods (i.e., TM II SOR, TSTMSOR, TSMSOR, TMSOR, and MSOR) extracted the following results. The CPU time and iteration steps calculated for the proposed methods are less than that of TSMSOR, TMSOR, and MSOR methods for \( \alpha = 1 \). Moreover, by increasing the size of the problem, this superiority remains. For \( \alpha \neq 1 \), when \( \alpha > 1 \) the new methods have a better performance than the three mentioned methods in terms of computing efficiency. Furthermore, the convergence region of the proposed methods (TM II SOR, TSTMSOR) is wider than other methods. Therefore, the results indicate, the proposed methods are efficient and effective ways to solve \( \text{LCP}(q, A) \). Note that based on the results recorded in the tables for the three examples, the TM II SOR method is superior to the TSTMSOR method in terms of CPU time and iteration steps.
Two-Step Two-Sweep Modulus-Based Matrix Splitting Iteration Method for LCP

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<tr>
<th>m</th>
<th>Method</th>
<th>α</th>
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<td>25</td>
</tr>
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</table>

### Acknowledgments

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References

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