Abstract. In this paper, we propose a novel PDE-based model for the multi-phase segmentation problem by using a complex version of Cahn-Hilliard equations. Specifically, we modify the original complex system of Cahn-Hilliard equations by adding the mean curvature term and the fitting term to the evolution of its real part, which helps to render a piecewisely constant function at the steady state. By applying the K-means method to this function, one could achieve the desired multiphase segmentation. To solve the proposed system of equations, a semi-implicit finite difference scheme is employed. Numerical experiments are presented to demonstrate the feasibility of the proposed model and compare our model with other related ones.

AMS subject classifications: 68U10, 65K10, 65N06
Key words: Image segmentation, Cahn–Hilliard equation, semi-implicit finite difference scheme.

1. Introduction

Image segmentation aims to partition an image domain into a few disjoint parts, each of which represents some meaningful object or background. It is a classical problem in image processing and has been extensively studied during the past few decades. Different mathematical approaches, including stochastic and deterministic methods, have been used to solve this problem. Among these methods, variational methods or PDE-based methods were widely utilized as they could be easily adapted to the complexities and flexibilities of real applications.

In the literature, many variational models have been proposed for solving the segmentation problem, including the well-known Mumford-Shah model [17], the classical...
snake model by Kass, Witkin, and Terzopoulos [12], the geodesic active contour model by Caselles, Kimmerl, and Sapiro [5], the Chan-Vese model [6], etc. Note that the three later models all deal with the problem of two-phase segmentation, that is, the resulting segmentation contour partitions an image domain into two parts, with one for objects and the other for background. Usually, there could exist multiple objects with distinct image intensities in a given image. Then the goal of segmentation is to separate all these objects, which leads to the multi-phase segmentation problem.

There also exist many multi-phase segmentation models in the literature. For instance, Vese and Chan [20] extended their original Chan-Vese model for the segmentation of multiple objects, i.e., the partition contains more than two regions. To this end, they considered multiple level set functions [18] and used the combination of the signs of these functions to label the resulting multiple regions. For instance, \( n \) level set functions can lead to a possibility of at most \( 2^n \) regions. Despite of the efficiency of representing regions using multiple level set functions, the numerical implementation is usually expensive due to the cost for each level set function. To overcome this issue, Chung and Vese [8] considered one implicit Lipschitz continuous function defined on the image domain and used it to separate different regions through its level lines. Lie et al. [14] introduced the piecewise constant level set method by assigning each of the constant values for a phase of the segmentation. In both methods, only one function is needed for multi-phase segmentation, which saves the computational effort remarkably. Later on, Jung et al. [11] proposed a multi-phase segmentation model that is based on the phase transition model of Modica and Mortola in material sciences. This model introduced a novel fitting term that uses the sinc-function to separate different phases.

Different from all the above approaches, Cai et al. [4] proposed a two-stage multi-phase segmentation model. Specifically, in the first stage, they solve a convex variant of the Mumford-Shah functional given as follows:

\[
\inf_g \left\{ \frac{\lambda}{2} \int_{\Omega} (f - Ag)^2 \, dx + \frac{\mu}{2} \int_{\Omega} |\nabla g|^2 \, dx + \int_{\Omega} |\nabla g| \, dx \right\},
\]

where \( f : \Omega \to \mathbb{R} \) is a given image defined on \( \Omega \subset \mathbb{R}^2 \), \( A \) represents a linear blurring operator or the identity operator, and \( \lambda, \mu > 0 \) are parameters. In this functional, the original length term of boundary in the Mumford-Shah functional is replaced by the total variation of the desirable clean function \( g \). Note that this new functional is convex, and its minimizer is unique. Then, as discussed in [4], the second stage is to segment the minimizer \( g \) into \( K \) phases (\( K \geq 2 \)) by using thresholds or by any clustering methods like the \( K \)-means method. This model possesses several merits:

1) when compared with the Mumford-Shah, the above model can be more easily handled both analytically and numerically;

2) the phase number \( K \) of segmentation can be chosen by users without re-calculating the minimizer \( g \).
Later on, Wu et al. [22] proposed a new nonconvex approximation of the Mumford-Shah model based on nonconvex \( \ell_p \) quasinorm regularizer for image segmentation which can extract more boundary information. In the meantime, Wu et al. [21] proposed the Adaptive Total Variation (ATV) model to adequately describe the local features of targets in images which have potential to compare with some deep learning method. Cai et al. [3] found the linkage between the piecewise constant Mumford-Shah model and the ROF model and therefore proposed the thresholded-ROF model to improve efficiency and effectiveness.

Recently, inspired by the work [2], we proposed a two-phase segmentation model by using a modified Cahn-Hilliard equation [23]. The motivation of using the Cahn-Hilliard equation originates from its attributes, that is, this equation describes the process of phase separation of a binary fluid. An interesting feature of this model lies in the fact that it can automatically interpolate missing contours along wide gaps in order to form meaningful object boundaries, which is often achieved by curvature based segmentation models [1, 25]. Moreover, when compared with those Euler-Lagrange equations associated with curvature dependent models, even though the Cahn-Hilliard equation is still of fourth-order, its highest order term is linear, which significantly reduces the hardness of designing effective numerical methods for solving the proposed model.

In this work, we intend to extend our work [23] to deal with the multi-phase segmentation problem. For this, we would like to employ the complex version of Cahn-Hilliard equation, which was introduced in [7, 9] for grayscale image inpainting. By using a complex version of Cahn-Hilliard equation, each phase of segmentation can be represented by the real part of the goal complex variable. To ensure that the real part takes on piecewise constant values over each phase, we propose adding a novel term originating from the total variation. Specifically, we incorporate a curvature term to the equation of the real part. Once the system of the complex version of Cahn-Hilliard equations is solved, as [4], we can apply clustering methods together a given phase number to determine the goal multi-phase segmentation. Therefore, as [4], one only needs to solve the system once, and the segmentation can be obtained by the choice of phase number.

The main contributions of this paper are in the following:

1. We propose a new PDE-based model for multi-phase image segmentation using the complex version of Cahn-Hilliard equation by adding the mean curvature term to its real part and also we give a priori estimate for solution \( u(x, t) \).

2. We apply a semi-implicit scheme to solve the proposed model and provide a stability analysis of the scheme.

The rest of this paper is organized as follows. We describe the proposed model in Section 2, a priori estimate is given in Section 3 and our algorithm is presented in Section 4, some numerical experiments are given in Section 5, and the conclusions follow in Section 6.
2. The proposed model

Before presenting our model for multi-phase segmentation, we recall some closely related works.

Bertozzi et al. [2] proposed using the Cahn-Hilliard equation for binary image inpainting. Specifically, they solved the following equation to steady state in order to interpolate the missing image information in the inpainting region:

\[
\frac{\partial u}{\partial t} = -\Delta \left( \epsilon \Delta u - \frac{1}{\epsilon} W'(u) \right) + \lambda \chi_{\Omega \setminus D}(f - u)
\]  

(2.1)

with the boundary conditions

\[
\frac{\partial u}{\partial \nu} = \frac{\partial \Delta u}{\partial \nu} = 0 \quad \text{on} \quad \Omega,
\]

where \( \nu \) represents the outer unit normal vector to \( \partial \Omega \). Here \( f \) is a given image defined on \( \Omega \), \( D \) is the inpainting region, and \( \lambda \) is a parameter. The above equation is not a gradient flow for some functional. In fact, the Cahn-Hilliard equation, i.e., the first part on the right hand side, is the \( H^{-1} \) gradient flow of the following energy:

\[
E_{\epsilon}(u) = \int_{\Omega} \left[ \epsilon |\nabla u|^2 + \frac{1}{\epsilon} W(u) \right] dx,
\]

(2.2)

where the parameter \( \epsilon > 0 \) is an interaction length, and \( W(u) = u^2(u - 1)^2 \) is a double well potential function. The second term is the usual \( L^2 \) gradient flow of the energy \( \frac{1}{2} \int_{\Omega \setminus D} (f - u)^2 dx \), which requires that the obtained image \( u \) be close to \( f \) in the region with known intensity information of \( f \).

Note that Bertozzi et al. model can handle the inpainting for binary images. To extend this model for generic images, Cherfils et al. [7] proposed using a complex version of the Cahn-Hilliard equation. For this, they considered the following free energy:

\[
\mathcal{E}(u) = \int_{\Omega} \left[ \frac{\epsilon}{2} |\nabla u|^2 + \frac{1}{4\epsilon} |u|^4 - \frac{1}{2\epsilon} |u|^2 \right] dx,
\]

(2.3)

or equivalently

\[
\mathcal{E}_1(u) = \int_{\Omega} \left[ \frac{\epsilon}{2} |\nabla u|^2 + \frac{1}{4\epsilon} (|u|^2 - 1)^2 \right] dx,
\]

(2.4)

where \( u = u_1 + iu_2 \) is a complex function defined on \( \Omega \), and \( |u| = \sqrt{u_1^2 + u_2^2} \). Standard calculation leads to

\[
\frac{\delta \mathcal{E}_1}{\delta u} = -\epsilon \Delta u + \frac{1}{\epsilon} (|u|^2 - 1) u,
\]

(2.5)

where \( \Delta u = [\Delta u_1, \Delta u_2]^T \). With this, Cherfils et al. [7] derived the complex version of Cahn-Hilliard equation

\[
\frac{\partial u}{\partial t} = -\Delta \left( \epsilon \Delta u - \frac{1}{\epsilon} (|u|^2 - 1) u \right),
\]

(2.6)
and proposed the following system of equations for generic image inpainting:

$$\frac{\partial u}{\partial t} = -\Delta \left( \epsilon \Delta u - \frac{1}{\epsilon} (|u|^2 - 1) u \right) + \lambda \chi_{\Omega \setminus D} (f - u)$$  \hspace{1cm} (2.7)$$

with the boundary conditions

$$\frac{\partial u}{\partial \nu} = \frac{\partial \Delta u}{\partial \nu} = 0 \quad \text{on} \quad \partial \Omega.$$

As discussed in [7], the double well potential $W(u) = u^2(u - 1)^2$ was replaced by $W(u) = (|u|^2 - 1)^2$, and thus its minima satisfy $|u| = 1$, that is, not merely $u = 0$ or $u = 1$, which helps to handle the inpainting for generic images. In fact, before applying the above models, any given image should be rescaled in $[0, 1]$ for its intensity.

In this work, we propose applying the complex version of Cahn-Hilliard for the multi-phase segmentation problem. To this end, besides the functional $E_1(u)$ in (2.4), we also consider the functional of $u_1$ as follows:

$$E_2(u_1) = \lambda_2 \int_{\Omega} |\nabla u_1| + \frac{\lambda_1}{2} \int_{\Omega} (f - u_1)^2,$$

which is the same as the well-known Rudin-Osher-Fatemi model [19]. The introduction of this term is based on the feature of the ROF model, that is, its minimizer prefers a piecewisely constant function. This piecewise function leads to the unfavorable feature, that is, the staircase effect, for image denoising. However, it turns out to be a merit for our segmentation problem, since a piecewisely constant function helps to partition an image domain into different regions. Indeed, the Chan-Vese model employs a binary function to approximate the given image function in order to achieve two-phase segmentation [6], and this piecewise constant function is constructed by using several level set functions in [20] for multi-phase segmentation. Moreover, in [14–16], the piecewise constant level set method was developed for multi-phase segmentation. All these models utilize piecewise constant functions for the purpose of image segmentation.

Based on the above discussion, we propose the following system of complex version of Cahn-Hilliard equations for the multi-phase segmentation:

\[
\begin{align*}
\begin{cases}
\quad u_{1,t} &= -\Delta \left( \epsilon_1 \Delta u_1 - \frac{1}{\epsilon_1} W'(u) \right) + \lambda_1 (f - u_1) + \lambda_2 \nabla \cdot \left( \frac{\nabla u_1}{|\nabla u_1|} \right), & \text{in } \Omega \times (0, T], \\
\quad u_{2,t} &= -\Delta \left( \epsilon_1 \Delta u_2 - \frac{1}{\epsilon_1} W'(u) \right), & \text{in } \Omega \times (0, T], \\
\quad \frac{\partial u}{\partial n} &= \frac{\partial \Delta u}{\partial n} = 0, & \text{on } \partial \Omega \times [0, T], \\
\quad u_1 &= f, \quad u_2 = \sqrt{1 - f^2}, & \text{on } \Omega \times \{t = 0\},
\end{cases}
\end{align*}
\]  \hspace{1cm} (2.9)
where \( f \) is the given image whose intensity is rescaled to be in \([0, 1]\) and \( T > 0 \). Moreover, \( \lambda_1, \lambda_2 > 0 \) are tuning parameters.

In the above system, the second equation keeps the same form as the complex version of Cahn-Hilliard equations (2.6), since \( u_2 \) is associated with \( u_1 \) through the potential \( W(u) \), while only \( u_1 \) will be used as the segmentation function. This is different from Cherfils et al. inpainting model (2.7), whose fitting term involves both the real and imaginary part of \( u = u_1 + iu_2 \).

Once the above system is solved to its steady state, the real part function \( u_1 \) can be used to segment the image domain \( \Omega \) into several parts. As a matter of fact, as discussed above, with the total variation term being involved in our model, the function \( u_1 \) assumes piecewise constant values over the domain. To obtain the final segmentation, just as in [4], one could use any clustering methods like K-means method [10].

3. A priori estimate

In this section, we give a priori bound estimate for the \( L^2 \)-norm of the solution \( u \), and we employ the similar technique as the one in [2]. For this, we denote \( \langle \cdot, \cdot \rangle \) as the usual \( L^2 \) product with associated norm \( \| \cdot \| \) and introduce a weak solution for the proposed system of Cahn-Hilliard type equations (2.9) as follows.

**Definition 3.1.** We say \( u = (u_1, u_2) \) is a weak solution of Eq. (2.9) if \( u \) satisfies the initial conditions described in Eq. (2.9) and the following also holds:

\[
\frac{d}{dt} \langle u_1, v \rangle + \langle \epsilon_1 \Delta u_1, \Delta v \rangle - \left\langle \frac{1}{\epsilon_1} W'(u), \Delta v \right\rangle = \langle \lambda_1 (f - u_1), v \rangle + \left\langle \lambda_2 \nabla \cdot \left( \nabla u_1 \left| \nabla u_1 \right| \right), v \right\rangle,
\]

\[
\frac{d}{dt} \langle u_2, v \rangle + \langle \epsilon_1 \Delta u_2, \Delta v \rangle - \left\langle \frac{1}{\epsilon_1} W'(u), \Delta v \right\rangle = 0
\]

for any \( v \in V \) with

\[
V = \left\{ \phi \in H^2(\Omega) \mid \frac{\partial \phi}{\partial \vec{n}} = \frac{\partial \Delta \phi}{\partial \vec{n}} = 0 \text{ on } \partial \Omega \right\}.
\]

**Lemma 3.1.** Given a weak solution \( u = u_1 + iu_2 \) as described above, there exist constants \( c_1, c_2 > 0 \), such that

\[
\|u(\cdot, t)\|_{L^2} \leq e^{c_2(\delta_1, \varepsilon_1)t} (\|u(\cdot, 0)\|_{L^2} + c_1 (\|\Omega\|, \lambda_1)t)
\]

for all \( 0 \leq t \leq T \) and \( T \geq 0 \).

**Proof.** Multiply the first two equations in (2.9) by \( u_1 \) and \( u_2 \) respectively, and integrate on \( \Omega \),

\[
\frac{1}{2} \int \frac{d}{dt} (u_1^2) \, dx = -\varepsilon_1 \int \Delta u_1^2 \, dx + \frac{1}{\varepsilon_1} \int W'(u) \Delta u_1 \, dx
\]
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\[
\frac{1}{2} \int \frac{d}{dt} (u_i^2) \, dx = -\varepsilon_1 \int \Delta u_i^2 \, dx + \frac{1}{\varepsilon_1} \int W'(u_i) \Delta u_i \, dx.
\]

Summing of above two equations, we get

\[
\frac{1}{2} \int \sum_{i=1}^{2} \frac{d}{dt} (u_i^2) \, dx
\]

\[
= -\varepsilon_1 \int \sum_{i=1}^{2} \Delta u_i^2 \, dx + \lambda_1 \int (f - u_1) u_1 \, dx - \lambda_2 \int |\nabla u_1| \, dx.
\]

(3.1)

For the second term in the right side in above equation, we have

\[
\lambda_1 \int (f - u_1) u_1 \, dx \leq \lambda_1 \|f\|_2 \|u_1\|_2 - \lambda_1 \|u_1\|_2^2 \leq \frac{\lambda_1}{2} \|f\|_2^2 - \frac{\lambda_1}{2} \|u_1\|_2^2.
\]

For the last term in Eq. (3.1),

\[
\frac{1}{\varepsilon_1} \int \left( \sum_{i=1}^{2} u_i^2 - 1 \right) \left( \sum_{i=1}^{2} u_i \Delta u_i \right) \, dx
\]

\[
= -\frac{1}{\varepsilon_1} \int \sum_{i=1}^{2} (3u_i^2 - 1) (\nabla u_i)^2 \, dx + \frac{1}{\varepsilon_1} \int (u_2^2 u_1 + u_1^2 u_2^2) \, dx
\]

\[
= -\frac{1}{\varepsilon_1} \int \sum_{i=1}^{2} (3u_i^2 - 1) (\nabla u_i)^2 \, dx - \frac{1}{\varepsilon_1} \int (\nabla u_1)^2 u_2^2 + (\nabla u_2)^2 u_1^2 \, dx
\]

\[
- \frac{4}{\varepsilon_1} \int u_1 u_2 \nabla u_1 \nabla u_2 \, dx.
\]

For the last term in the above equation,

\[
-\frac{4}{\varepsilon_1} \int u_1 u_2 \nabla u_1 \nabla u_2 \, dx \leq \frac{4}{\varepsilon_1} \|u_1 \nabla u_1\|_2 \|u_2 \nabla u_2\|_2 \leq \frac{2}{\varepsilon_1} \sum_{i=1}^{2} \|u_i \nabla u_i\|_2^2.
\]

Furthermore,

\[
-\frac{1}{\varepsilon_1} \int \sum_{i=1}^{2} (3u_i^2 - 1) (\nabla u_i)^2 \, dx = -\frac{3}{\varepsilon_1} \int \sum_{i=1}^{2} u_i^2 (\nabla u_i)^2 \, dx + \frac{1}{\varepsilon_1} \int \sum_{i=1}^{2} (\nabla u_i)^2 \, dx.
\]

For \(\frac{1}{\varepsilon_1} \int (\nabla u_i)^2 \, dx\), by interpolation inequality, we obtain

\[
\int_{\Omega} |\nabla u_i|^2 \, dx \leq \delta_1 \int_{\Omega} (\Delta u_i)^2 \, dx + \frac{1}{\delta_1} \int_{\Omega} u_i^2 \, dx, \quad \forall \delta_1 > 0.
\]
Finally, put everything together, we get

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} u dx \leq -\varepsilon_1 \int_{\Omega} \sum_{i=1}^{2} \Delta u_i^2 dx - \frac{1}{\varepsilon_1} \int_{\Omega} u_1^2 (\nabla u_i)^2 dx$$

$$- \frac{1}{\varepsilon_1} \int_{\Omega} (\nabla u_1)^2 u_2^2 + (\nabla u_2)^2 u_2^2 dx$$

$$+ \frac{1}{\varepsilon_1} \left( \delta_1 \int_{\Omega} (\Delta u_1)^2 dx + \frac{1}{\delta_1} \int_{\Omega} u_1^2 dx \right)$$

$$+ \frac{\lambda_1}{2} ||f||_2^2 - \frac{\lambda_1}{2} ||u_1||_2^2 - \lambda_2 \int |\nabla u_1| dx. \quad (3.2)$$

If we choose $\delta_1$ small enough, such that

$$\varepsilon_1 > \frac{\delta_1}{\varepsilon_1}, \quad \text{i.e.} \quad \delta_1 < \varepsilon_1^2,$$

then we have,

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} u dx \leq c_1 (||\Omega||, \lambda_1) + c_2 (\delta_1, \varepsilon_1) ||u||^2.$$

Thus the differential form of Gronwall’s inequality yields the estimate

$$\|u(\cdot,t)\|_{L^2} \leq e^{c_2(\delta_1,\varepsilon_1) t} (\|u(\cdot,0)\|_{L^2} + c_1 (||\Omega||, \lambda_1) t).$$

From Eq. (3.2) and above estimation, integrate from 0 to $T$ we have

$$\max_{0 \leq t \leq T} \|u\|^2_{L^2} + \|u\|^2_{L^2(0,T;H^1_0(\Omega))} \leq \tilde{C}_4 \|u_0\|^2_{L^2} + \tilde{C}_5 T.$$

The proof is complete. $\square$

The existence result can be proved similarly as [23, Theorem 3.2] using the above estimation.

### 4. Numerical algorithm

In this section, the numerical method for our model (2.9) will be introduced. Instead of solving the system of fourth-order equations, as in [23], we introduce two auxiliary functions $v_1$ and $v_2$, and then rewrite our model (2.9) as follows:

$$\begin{aligned}
&u_{1,t} = -\Delta v_1 + \lambda_1 (f - u_1) + \lambda_2 \nabla \cdot \left( \frac{\nabla u_1}{|\nabla u_1|} \right), \\
v_1 = \varepsilon_1 \Delta u_1 - \frac{1}{\varepsilon_1} W'(u), \\
u_{2,t} = -\Delta v_2, \\
v_2 = \varepsilon_1 \Delta u_2 - \frac{1}{\varepsilon_1} W'(u).
\end{aligned}$$
To solve the new system, we employ semi-implicit schemes and thus discretize it as follows:

\[
\begin{align*}
\frac{u_1^{n+1} - u_1^n}{\tau} &= -\Delta v_1^{n+1} + \lambda_1 (f - u_1^{n+1}) + \lambda_2 \nabla \cdot \left( \frac{\nabla u_1^{n+1}}{|\nabla u_1^n|} \right), \\
v_1^{n+1} &= \varepsilon_1 \Delta u_1^{n+1} - \frac{1}{\varepsilon_1} u_1^{n+1} ((u_1^n)^2 + (u_2^n)^2 - 1), \\
\frac{u_2^{n+1} - u_2^n}{\tau} &= -\Delta v_2^{n+1}, \\
v_2^{n+1} &= \varepsilon_1 \Delta u_2^{n+1} - \frac{1}{\varepsilon_1} u_2^{n+1} ((u_1^n)^2 + (u_2^n)^2 - 1).
\end{align*}
\]

Due to the nature of image processing problem, we can set the \( h = 1 \) to be the mesh size in space, and therefore

\[
\Delta u_{i,j} = (u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j}).
\]

And for the mean curvature term,

\[
\nabla \cdot \left( \frac{\nabla u}{|\nabla u|} \right) = \frac{\partial}{\partial x} \left( \frac{u_x}{|\nabla u|} \right) + \frac{\partial}{\partial y} \left( \frac{u_y}{|\nabla u|} \right).
\]

So we can discrete it as

\[
\nabla \cdot \left( \frac{\nabla u}{|\nabla u|} \right)_{i,j} = \left( \frac{u_x}{|\nabla u|} \right)_{i+\frac{1}{2},j} - \left( \frac{u_x}{|\nabla u|} \right)_{i-\frac{1}{2},j} + \left( \frac{u_y}{|\nabla u|} \right)_{i,j+\frac{1}{2}} - \left( \frac{u_y}{|\nabla u|} \right)_{i,j-\frac{1}{2}}
\]

with

\[
\left( \frac{u_x}{|\nabla u|} \right)_{i+\frac{1}{2},j} = \frac{u_{i+1,j} - u_{i,j}}{\sqrt{\delta + (u_{i+1,j} - u_{i,j})^2 + (u_{i+1,j+1} + u_{i,j+1} - u_{i+1,j-1} - u_{i-1,j})^2/16}}.
\]

In the above equation, the \( \delta \) in the denominator is used to avoid the numerical singularity (we use \( \delta = 10^{-10} \) for all the experiments in this paper).

In terms of the Neumann boundary condition, we would like to use virtual point such as \( u_{-1,j} \) to deal with. Note that the second order Neumann boundary condition is equal to the zero order condition for variable \( v = \Delta u \), we can get \( u_{-1,j} = u_{1,j} \), \( v_{-1,j} = v_{1,j} \) on the left side which represent the zero order and second order Neumann boundary condition respectively, and the three other sides can be obtained similarly.

**Remark 4.1.** If we apply explicit schemes such as prediction-correction methods, the stability in time requires us to set a small time step which is around the \( 10^{-5} \) level. That would slow down the evolution of the system, while with a semi-implicit scheme, one could choose a time step on \( 10^{-2} \) level.

**Remark 4.2.** As for the nonlinear terms in the system, a fully implicit scheme brings about two nonlinear equations which would cost much more computing resource on solving the model.
Remark 4.3. Some high order finite difference methods, such as Crank-Nicolson scheme, would impact the convergence of numerical solution like spurious oscillation.

Then we present the stability analysis of the proposed numerical scheme.

Theorem 4.1 (Conditional stability). Under the assumption that $W''(u^{k-1}) \leq K$, we have the solution sequence $u^k$ is bounded on a finite time interval $[0, T]$, for $0 < \tau \leq \frac{\omega^2}{(K-2)^2}$. In particular for $k\tau ≤ T$ being fixed, there exist constants $M, N$ and a constant $C$ depending only on $\Omega, f$, and $\lambda_1$ such that the following estimate holds:

$$\|u^k\|^2 + \frac{N\tau}{1 - \tau M} \|\Delta u^k\|^2 \leq 2^{2MT} \left(\|u^0\|^2 + \frac{N\tau}{1 - \tau M} \|\Delta u^0\|^2 + 2TC(\Omega, f, \lambda_1)\right).$$

Proof. Multiplying by $u^{k+1}_1$ on the real part sides and integrating over $\Omega$, we get

$$\frac{1}{\tau} (\|u^{k+1}_1\|^2 - \langle u^k_1, u^{k+1}_1 \rangle) + \epsilon \|\Delta u^{k+1}_1\|^2$$

$$= \frac{1}{\epsilon} \langle u^{k+1}_1 ((u^k_1)^2 + (u^k_2)^2 - 1), \Delta u^{n+1}_1 \rangle + \lambda_1 \langle f - u^{k+1}_1, u^{k+1}_1 \rangle$$

$$+ \lambda_2 \langle \nabla \cdot \left( \frac{\nabla u^{k+1}_1}{|\nabla u^{k}_1|} \right), u^{k+1}_1 \rangle.$$

By Young’s inequality and integration by part, we have

$$\frac{1}{2\tau} \left(\|u^{k+1}_1\|^2 - \|u^k_1\|^2\right) + \epsilon \|\Delta u^{k+1}_1\|^2$$

$$\leq \frac{\lambda_1}{2} \|f\|^2 - \frac{\lambda_1}{2} \|u^{k+1}_1\|^2 - \lambda_2 \left(\frac{\nabla u^{k+1}_1}{|\nabla u^{k}_1|}, \nabla u^{k+1}_1 \right)$$

$$+ \frac{1}{\epsilon} \langle u^{k+1}_1 ((u^k_1)^2 + (u^k_2)^2 - 1), \Delta u^{n+1}_1 \rangle.$$

Note that

$$\langle \frac{\nabla u^{k+1}_1}{|\nabla u^{k}_1|}, \nabla u^{k+1}_1 \rangle \geq 0,$$

and by the assumption that $W''(u) \leq K$ (which is actually equivalent to $|u|^2 \leq K^2 \frac{1}{3}$), we have

$$\frac{1}{\epsilon} \langle u^{k+1}_1 ((u^k_1)^2 + (u^k_2)^2 - 1), \Delta u^{n+1}_1 \rangle \leq \frac{1}{\epsilon} \|u^{k+1}_1\| \cdot \|\Delta u^{k+1}_1\| \cdot \frac{K - 2}{3} \leq K - 2 \frac{2}{6\epsilon} \left(\delta \|u^{k+1}_1\|^2 + \frac{\lambda_1}{\delta} \|\Delta u^{k+1}_1\|^2\right).$$

And then reordering the terms, we have

$$\left(\frac{1}{2\tau} - \frac{(K - 2)\delta}{6\epsilon} + \frac{\lambda_1}{2}\right) \|u^{k+1}_1\|^2 + \left(\epsilon - \frac{K - 2}{6\epsilon\delta}\right) \|\Delta u^{k+1}_1\|^2$$

$$\leq \frac{1}{2\tau} \|u^k_1\|^2 + C(\Omega, f, \lambda_1).$$
Note that the imaginary part equation $u_2$ is similar to the real part by putting $\lambda_1$ and $\lambda_2$ equal to zero. So summing up both sides and multiply both side with $2\tau$, we get

$$
\left(1 - \tau \frac{(K - 2)\delta}{3\epsilon}\right) \|u^{k+1}\|^2 + \tau \left(2\epsilon - \frac{K - 2}{3\epsilon}\delta\right) \|\Delta u^{k+1}\|^2 \leq \|u^k\|^2 + 2\tau C(\Omega, f, \lambda_1),
$$

while

$$
\|u^k\|^2 = \|u_1^k\|^2 + \|u_2^k\|^2,
$$

we set

$$
M := \frac{(K - 2)\delta}{3\epsilon}, \quad N := 2\epsilon - \frac{K - 2}{3\epsilon}\delta,
$$

when $\tau \leq \frac{\epsilon a}{(K - 2)\delta}$, we can chose suitable $\delta$ so that $\tau M < \frac{1}{2}$ and $N > 0$. Dividing by $(1 - \tau M)$, we get

$$
\|u^{k+1}\|^2 + \frac{N\tau}{1 - \tau M} \|\Delta u^{k+1}\|^2 \leq \frac{1}{1 - \tau M} \left(\|u^k\|^2 + \frac{N\tau}{1 - \tau M} \|\Delta u^k\|^2\right) + 2\tau \frac{1}{1 - \tau M} C(\Omega, f, \lambda_1).
$$

By induction on $k$, it follows that:

$$
\|u^{k+1}\|^2 + \frac{N\tau}{1 - \tau M} \|\Delta u^{k+1}\|^2 \leq \left(\frac{1}{1 - \tau M}\right)^k \left(\|u^0\|^2 + \frac{N\tau}{1 - \tau M} \|\Delta u^0\|^2\right) + \sum_{i=0}^{k-1} \left(\frac{1}{1 - \tau M}\right)^i 2\tau C(\Omega, f, \lambda_1).
$$

Note that $(1 - x)^{-1/2}$ is monotonically increasing in $(0, 1)$ and $\tau M < \frac{1}{2}$, therefore we obtain for $k\tau \leq T$,

$$
\|u^{k+1}\|^2 + \frac{N\tau}{1 - \tau M} \|\Delta u^{k+1}\|^2 \leq \left(\frac{1}{1 - \tau M}\right)^{Mk\tau} \left(\|u^0\|^2 + \frac{N\tau}{1 - \tau M} \|\Delta u^0\|^2 + 2TC(\Omega, f, \lambda_1)\right) \leq 2^{2MT} \left(\|u^0\|^2 + \frac{N\tau}{1 - \tau M} \|\Delta u^0\|^2 + 2TC(\Omega, f, \lambda_1)\right).
$$

The proof is complete. 

\[\square\]

**Remark 4.4.** Since we are interested in bounded solutions $u^{k-1}$ of the numerical scheme, it is natural to assume $W''(u^{k-1}) \leq K$ for a constant $K$ which can be chosen arbitrarily large.
5. Numerical experiments

In this section, we present some numerical experiments by applying our complex modified Cahn-Hilliard model on various synthetic and real images, and also compare our model with some related models for multi-phase segmentation, including the models by Cai et al. [4], Li et al. [13], and Yuan et al. [24]. For all the numerical experiments, to determine whether the steady-state solutions are approached, we use the following stopping criterion:

$$\frac{\|u^k - u^{k-1}\|_2}{\|u^{k-1}\|_2} < tol$$

with the $tol = 10^{-5}$ and the time step $\tau = 0.02$. Once the steady-state solutions are approached, as in [4], we apply the K-means method for the real part function $u_1$ to get the multi-phase segmentation.

Before presenting the numerical results, we first discuss the parameters used in our experiments. Note that in our proposed model, there are three parameters: $\varepsilon_1, \lambda_1, \lambda_2$. The parameter $\varepsilon_1$ mainly determines the diffusion of $u$, the parameter $\lambda_1$ controls how closely the real image $f$ and the solution $u$ could be, while the parameter $\lambda_2$ helps render a piecewise constant function $u_1$, which is crucial to separate different phases.

Note that we are going to deal with a time-dependent equations for our model. The initial values of $u(x, 0)$ need to be set up. In our experiments, we let $u_1(x, 0)$ (the real part of $u(x, 0)$) be the given image which is scaled in the interval $[0, 1]$. Furthermore, we assume that the imaginary part of the initial datum is given by

$$u_2(x, 0) = \text{Im}[u(x, 0)] = \sqrt{1 - (u_1(x, 0))^2}.$$

**Example 5.1.** To see the performance of our model, we first conduct an experiment for a synthetic image as in [4], where different geometric shapes present in different gray intensities as shown in Fig. 1. This image is contaminated by Gaussian noise with zero mean and variance 0.03. One can easily see that there exist four phases in this image.

We compare our results with models in [4, 13, 21, 22, 24]. These plots show that almost all these models are able to segment the four phases successfully. However, the segmentation results differ in several aspects. First, our model could keep those boundaries very well, even for the noisy image, while the result obtained in [24] presents dented parts along those boundaries. Second, even though our segmentation result is very similar to the result in [4], the two segmentation functions exhibit different features. To see this, as shown Fig. 2, we present the comparison of slices for the two functions. These slice plots show that the segmentation function by our model tends to be piecewise constant over the domain, while the function in [4] fails to assume such a feature. And results given in [21, 22] both work very well for this example as our model. To see this clearly, we present in Fig. 3 with the three-dimensional plots of the given noisy image and the segmentation function $u_1$ by our model. In fact, one could
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Figure 1: The first row presents the clean image, noised image and the segmentation result by the model in [13]. The second row lists the results by models in [4, 22, 24]. The third row shows the results by the model in [21] and results by our model. The parameters used in this experiment for our model are as follows: \( \epsilon_1 = 0.25, \lambda_1 = 50, \lambda_2 = 10 \).

expect to obtain a more desirable segmentation result by using clustering methods for a piecewise constant function than for other functions.

**Example 5.2.** We then consider an MRI image in Fig. 4. This image also depicts four phases, that is, the dark background, the less dark region in the middle, the grey region, and the white region. Usually the identification of the border between the gray and white part provides crucial information in MRI image analysis.

For this experiment, we also compare our results with those obtained by the above five models. The plots show that our result is similar to the ones by all those models.
except model in [24], which combines the grey and white regions into one. Moreover, model in [13] needs to determine the phase number before implementing it, and for any other phase number, one has to run the model again, while our model, just as model in [4], only generates the segmentation function once and the segmentation result is determined by the K-means method with a choice of phase number. Models in [21, 22] could catch the edge and the grey regions well without omitting little grey dot regions. To further compare model in [4] and our model, in Fig. 2, we again present the slices, which show that both models could keep the locations of edges while our model is more inclined to produce piecewise constant segmentation function, which helps locate segmentation boundaries.

The above two examples demonstrate that our model is able to segment multiple phases for a given image. In Fig. 5, we present the plot of relative error (in log-scale) vs iteration number. This plot shows that the proposed iterative process is convergent and our model could yield desirable segmentation results with limited iterations of solving the proposed system of equations.

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Figure 4: The first row presents the given image and the segmentation results by models in [4, 13, 24]. The second row lists the results by models in [21, 22] and our model. The parameters used in this experiment for our model are as follows: $\epsilon_1 = 0.25$, $\lambda_1 = 35$, $\lambda_2 = 5$.

Example 5.3. To further illustrate the performance of our model, we consider three more real images in Figs. 6, 9 and 10. These real images are more complicated than those previous ones. In fact, in these images, the boundaries that separate different phases might not be well defined and even in the same phase, the image intensity could vary in a large amount. These facts raise a challenging problem of multiphase segmentation. Even for human vision, one could not claim the phase number affirmatively for these images.

Figure 5: The plot for the relative error vs iteration number for the Example 5.2 in Fig. 4.
Figure 6: The first row presents the clean image and noised image. The second and third row lists results by our model. The parameters used in this experiment for our model are as follows: $\varepsilon_1 = 0.25$, $\lambda_1 = 15$, $\lambda_2 = 10$.

For these images, we first apply the proposed model to obtain the segmentation function $u_1$ and then employ the K-means method to get the final segmentation. In Figs. 6-10, we present the segmentation function $u_1$ and the segmentation results by using the K-means method with different phase numbers. One could see that both the results for four-phase and five-phase are reasonable for human vision.

**Example 5.4.** Just as the model in [23], our model could also interpolate missing boundaries in order to form meaningful segmentation results. In Fig. 12, we consider
Figure 7: The first row presents the cut line on noised/final image and relative error curve of our numerical scheme. The second row lists the three-dimensional results before our model and after our model.

Figure 8: The first row presents the cut line on noised/final image and relative error curve of our numerical scheme. The second row lists the three-dimensional results before our model and after our model.
Figure 9: The first row presents the clean image and noised image. The second row lists results by our model. The parameters used in this experiment for our model are as follows: $\varepsilon_1 = 0.25$, $\lambda_1 = 10$, $\lambda_2 = 10$.

In a conclusion, those numerical examples show the efficiency of our model for the multi-phase segmentation problem. In fact, these experiments illustrate that our model privileges a piecewise constant function to approximate a given image while also main-
Figure 10: The first row presents the clean image and noised image. The second row lists results by our model. The parameters used in this experiment for our model are as follows: $\varepsilon_1 = 0.25$, $\lambda_1 = 10$, $\lambda_2 = 10$.

contains the location of edges regardless of noise. All these merits help generate desirable segmentation results by applying clustering algorithms like the K-means method for the obtained segmentation function.

In this work, we solve the above system of $u$ and $v$ accurately using linear sparse LU system solver. In fact, one could also solve it with other fast solvers like BiCGSTAB or GMRES with suitable preconditioner and initial guess to get some inexact solution for each iteration, which could help save the cost of restoring and transferring data especially for images of large sizes.
Figure 11: The first row presents the cut line on noised/final image and relative error curve of our numerical scheme. The second row lists the three-dimensional results before our model and after our model.

Figure 12: A given image with incomplete letters “Tsinghua” in different phase and the segmentation result by our model.

6. Conclusions

In this work, we propose a new multi-phase segmentation model by using a complex version of Cahn-Hilliard equations. In order to render piecewise constant segmentation functions, we add a mean curvature term for the real part of the goal complex function to the standard complex version of Cahn-Hilliard equations. We also give a priori
estimate for the solution of our model. To solve the proposed system of equations, we employ a semi-implicit scheme that can be implemented easily. The numerical experiments presented in this paper demonstrate that our model is able to segment objects from different phases.

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References


